Math behind the Rubik's Cube

Group 8-32

Project Work Written Report

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1. **Introduction**

The Rubik's Cube was originally invented by professor of architecture Erno Rubik. Over the past few years, the cube has become a very popular toy for people of all ages to play with. The cube consists of 6 faces and 3 layers, with 9 small cubes on each face.

While cube solvers are able to solve the cube solely by memorising algorithms, in this project, we look more in-depth into how math is involved in solving the Rubik's Cube, to give us cube solvers a better understanding of how and why the Rubik's Cube works, so that we know the reasons why the memorised algorithms can work.

a) **Objective of this Project**

To find out how math is involved in solving the Rubik's Cube

b) **Research Questions**

When we first started out on this project, we wanted to find out the following : What are the Math Concepts behind the Rubik's Cube? How is the Rubik's Cube solved? How would changing a certain step/permutation in solving the Rubik's cube affect the time required to solve the cube? Why?

2. **Literature Review**

The math behind the Rubik's Cube involves things like Group theorems, as well as different types of moves including Commutative and Non-Commutative ones, but we will focus on these two in this project. 4 axioms are involved in the solving of the Rubik's Cube – Closure, Associativity, Identity and Inverse.

- Closure : When a move is performed on the cube, another move will complement that move, causing small cubes to be put in place.
- Associativity : e.g. (R) RR = RR(R) performing either move first will give the same result
- Identity : Crucial during solving different layers – Taking the White Face as a reference to ensure moves are performed correctly
- Inverse – Performing moves backward (e.g. R inverse -> R')

3. **Methodology**

a) **Math concepts of the Cube**

i) **Functions :**

- Input, Output

Some types of Functions :

$X^2/X^3 + 1$

So in a Rubik's cube : R' U' L', the apostrophe is the input while the letters
Are the function names, therefore R + apostrophe = R' which is the output

R function name (‘) input = R’ output

ii) Commutative moves and non-commutative moves

- Two moves that do not involve the same small cubes, e.g. U D = U’ D’
  $UDU^{-1}D^{-1} = 1$ -> Commutative, UD = DU
- Some non-commutative moves : RU, UL etc. because these moves involve the same small cubes
- In U^{-1}/D^{-1} , ‘-1’ is the inverse function, U/D is the function name and U^{-1}/D^{-1} is the output
- Some situations that commutative moves are used include solving the first layer of the cube etc.

iii) Theorems About Groups (Group Theorems)

Scenario : Let e be the identity element, a and b are moves on the cube

If ab = e, then $a = b^{-1}$

Identity element – element that leaves any of the other elements unchanged when combined with them.

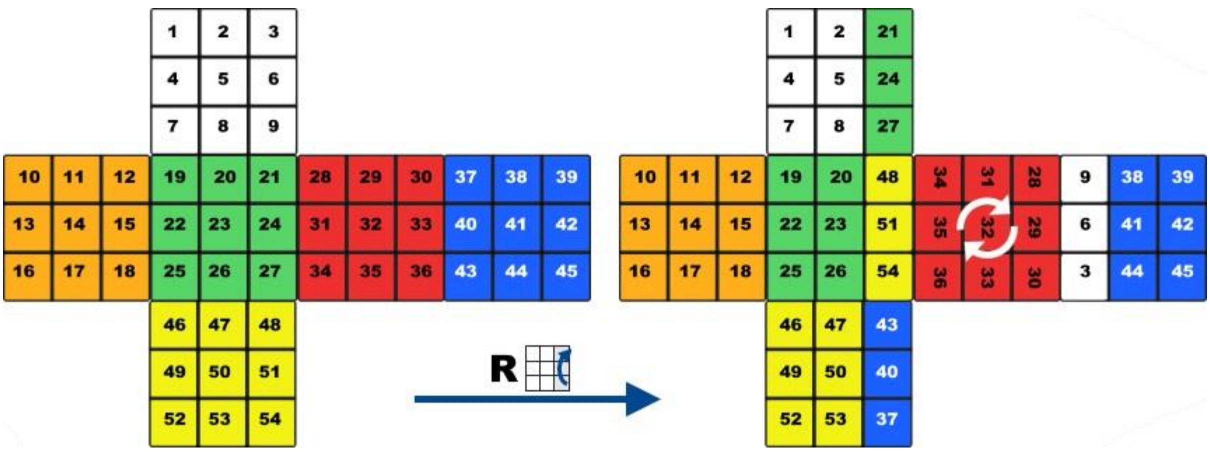

Diagram 1
Image taken from https://ruwix.com/the-rubiks-cube/mathematics-of-the-rubiks-cube-permutation-group/

b) How math is used in solving the rubik’s cube

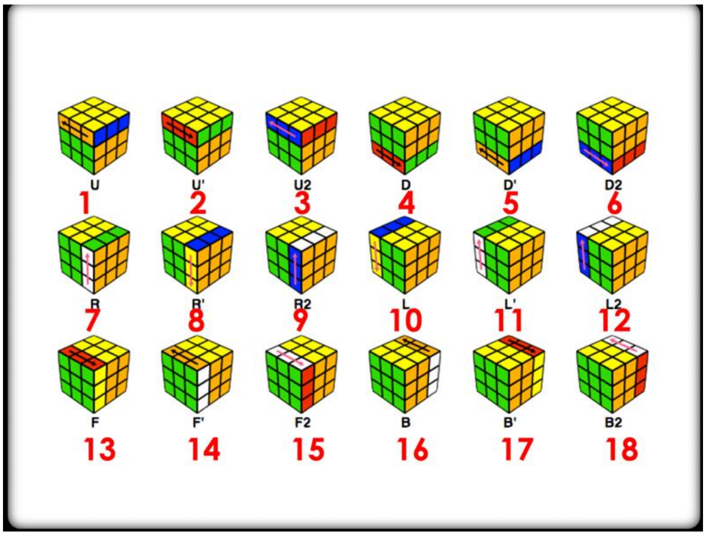

Diagram 2

Two main algorithms that are used when solving the rubik's cube are called the left and right triggers. The permutations of these are as such:

- **Left Trigger : L'U'L (11, 2, 10)**
- **Right Trigger : R U R' (7, 1, 8)**

When solving the first layer of the cube, our aim is also to complete the white face such that it becomes the identity element which we can take reference to when solving other faces

Group Theorems (applied to parts of the left and right triggers)

Group Theorems are applied to many moves in a Rubik's cube. As mentioned earlier, this is an example of a Group Theorem :

If ab = e, then a = b^{-1}

From the above statement, we can see that in 'a = b^{-1}', an inverse function is applied to 'b'. This means that b must be done in the inverse.

For the left trigger : If L'U' = e, then L' = U

This above statement means that if performing the moves L' then U' , L'U' becomes the identity element (e). Therefore, to get L', the move U'^{-1} (when simplified becomes U) has to be performed to the identity element to get L'.

How does this work?

The above is a Group Theorem, and combining a move with the identity element leaves other elements unchanged. Therefore, taking 'e' as the identity element, when the move U is combined with the identity element, the element L' is unchanged. Causing this theorem to work.

Solving the First Layer

Type of moves involved in this process : Non-Commutative moves

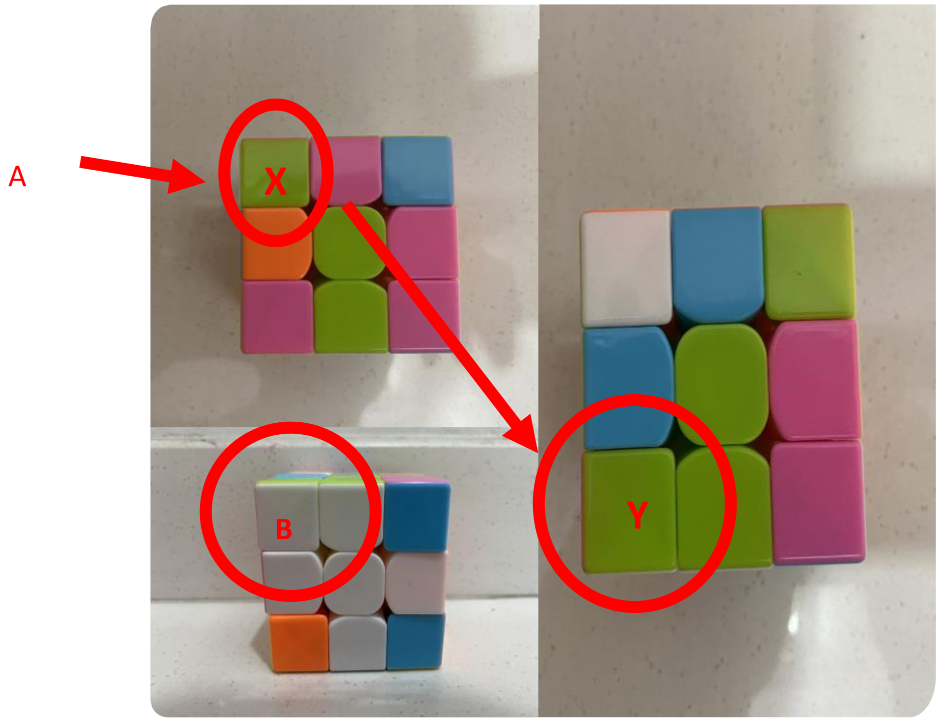

Diagram 3

This is a non-commutative move, where the left trigger is used to perform two acts. The aim is to move the green face of a corner piece (a piece with 3 faces) from position X to position Y. Do note that position A cannot be seen in the picture, but it is on the left of position X.

Why can this move work?

This move is non-commutative. As such, the moves that are performed, namely L', U' and L, involve the movement of common pieces of the cube.

We can thus say that $L'U'L \neq LUL'$ ***or*** $L'U'L \neq L'^{-1}U'^{-1}L^{-1}$

Solving the Second Layer

Key objective: Moving pieces from the top layer (yellow) to the second layer.

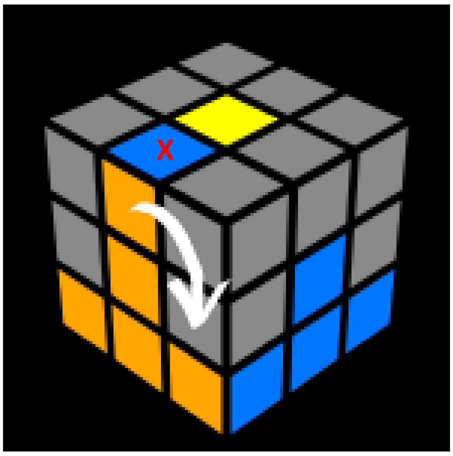

Diagram 4

There are two algorithms to rotate the piece to the second layer, in either right or left.

Right algorithm:

U R U’ R’ U’ F’ U F

Left algorithm:

U’ L’ U L U F U’ F’

We must focus on the colour of the piece marked 'X' to solve the second layer. These algorithms help to rotate the piece to the correct position for us to solve it next time. The orange piece is placed on top of the inverted 'T' shape as shown above.

Why does this work?

Over here, 3 out of 4 axioms are being utilised.
Let’s analyse the left and right algorithm.

Left Algorithm : **U’ L’ U L U F U’ F’**
Right Algorithm : **U R U’ R’ U’ F’ U F**

There is a perfect example of how the axiom ‘closure’ is used. In the Left algorithm…

U’ L’ | U L | U F | U’ F’

Diagram 5

This algorithm can be divided into 4 sets of small algorithms - each set is closed with the permutation L', L, F or F'. Each set contributes places the intended pieces in the right position because they are non-commutative moves. It is able to do so due to the use of the axiom 'closure' , meaning when a move is made, there must be another move to complement it to 'close' the algorithm and put the piece in place.

We shall look at how two more axioms, ‘identity’ and ‘inverse’ are involved here. When solving the second layer, the white face is taken as a reference, and faces the bottom, ensuring the permutations are always performed on the correct pieces. This makes it the identity, where cube solvers can tell if their moves are right or wrong by looking at whether the white face has any pieces out of place. UL has the input removed, making the output inverse of U’L’. Similarly, for U F, U’F’ is the inverse due to the addition of the input.

Solving the third layer (After solving the yellow cross) :

Algorithm used:

R U R' U R U2 R' U

The main goal of this stage is to match the colours of the edge pieces of each face in the top layer with that of the centre piece, to solve the cube. At this stage, you need to rotate around the top layer trying to find two edges which have to be switched. Thus, we execute the following algorithm to switch the edge pieces accordingly and solve the Rubik’s cube:

F2 U R’ L F2 L’ R U F’2 U

Now, why does this work?

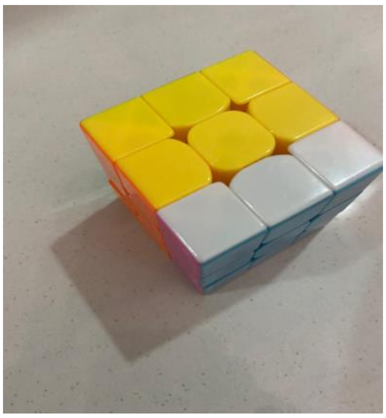

Diagram 6

From the above algorithm, we can see situations where non-commutative moves are used. For example, 'UR'' or 'R U', as UR is not = URU-1R-1. Also, using our first concept, on commutative and non-commutative moves, we can see in this algorithm that there is both use of non-commutative moves and commutative moves. As the permutation 'U' ensures that the edge piece rotates to another face, 2 moves are required to re-position the edge pieces. The function of a commutative move is to rotate the face which the small cube is positioned on, which leads us to the very last step in solving the rubik's cube – performing a commutative move position all small cubes accordingly.

Though it may look as simple as turning one layer of the cube and seem like there is no mathematical meaning to this, the fact is that such a move works due to it being a commutative move – L R= L R L^{-1} R^{-1}. Thus, this summarises how group theorems, axioms, commutative and non-commutative moves are used in solving the rubik's cube.

4. Results

Research Question 1 : What are the Math Concepts behind the Rubik's Cube?
In this project, we have covered how group theorems, axioms, functions, commutative and non-commutative moves, are involved in the solving of this rubik's cube.

Research Question 2 : How is the Rubik's Cube solved?
The Rubik's cube is solved using different permutations, which determine whether each move is commutative or non-commutative move. The **only** commutative moves in solving this rubik's cube are, UD, DU, LR, RL and their inverses, as they do not involve the same small cubes.

Research Question 3 : How would changing a certain step/permutation in solving the Rubik's cube affect the time required to solve the cube? Why?
Let's use solving the first layer of the cube as an example. If we think about everything we know, we will all understand that when the cube is solved, each face can only have one colour. Therefore, although the cube is still solvable if X does not have the same colour as the center piece, ensuring both X and the center piece of the face have the same colour will decrease the amount of time required to solve the cube, because we will not need to turn the first layer again after the white face is solved to match the correct colours to the corresponding centres of each face.

5. Conclusion

The Rubik's cube is solved by layer. At the end of solving each face, for example, in the earlier part of this report the white face, small cubes on the other faces must at least have one complete layer of the same colour to continue to allow the cube to be solved. Though the white face is solved, the blue and orange pieces are already in place for the first layer, allowing the second layer to be solved.

The Rubik's Cube is solved through Commutative and Non-Commutative Moves with Axioms and Group Theorems , layer by layer.

Diagram 7

6. Limitations and Extensions

Limitations

During the process of this project, reading took up fare bit of time as we were new to the concepts in the cube. Also, some of us did not know how to solve cube. Some of us had busier schedules, thus meeting at the same time was a bit of an issue

Extensions

This project can be extended further into discovering how other math concepts can be involved in solving of the cube, as well as expanding the project based on our 3rd research question, whereby we discover how other factors can affect how much time is required to solve the rubik's cube.

References

Math behind the solving of the rubik's cube(University of California, Berkeley)
Link :
https://math.berkeley.edu/~hutching/rubik.pdf

Group Theory 101 – How to play the Rubik's cube like a piano
(Staff, Michael)
https://www.youtube.com/watch?v=FW2Hvs5WaRY

Mathematics of the Rubik's Cube - Permutation Group, Ruwix Twisty Puzzle Wiki
https://ruwix.com/the-rubiks-cube/mathematics-of-the-rubiks-cube-permutation-group/

Rubik's Cube: How to Read Algorithms (Full Notation Guide), J Perm, published on youtube on 20 Apr 2019
https://www.youtube.com/watch?v=24eHm4ri8WM

Commutators and conjugates of the Rubik's Cube, Ruwix Twisty Puzzle Wiki
https://ruwix.com/the-rubiks-cube/commutators-conjugates/

Solving the last layer of the cube – yellow edges, Ruwix Twisty Puzzle Wiki
https://ruwix.com/the-rubiks-cube/how-to-solve-the-rubiks-cube-beginners-method/step-5-swap-yellow-edges