The Beautiful Math Behind Unit Fractions

G8-28 Project Work 2021

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Table of Contents

1. Introduction

1.1 Introduction to our Project Idea

Hi! As you can already tell, our project idea for the year revolves around a part of Mathematics known as Unit Fractions. A unit fractionis basically just a fraction that fulfils all these conditions we mention here.

~ It has to be a rational number of course

~ Must have its numerator be 1 and nothing else

~ Its denominator must be a positive integer (not0 or negative)

~ And therefore would be a reciprocal of a positive integer

Here are some examples and counterexamples of unit fractions.

Unit Fractions: $\frac{1}{2}, \frac{1}{340}, \frac{1}{10000}$

Non Unit Fractions: $\frac{6}{9}, \frac{3}{5}, \frac{-3}{8}$

1.2 Our Rationale

Firstly, to answer why we have chosen this very specific topics of Unit Fractions. We chose it due to its false simplicity and hidden complexity.

At first glance, this might seem like primary school maths, with 1 over something and that's it, compared to the various other complex fields of mathematics we cannot begin to understand, but as you start to dig deeper into the subject, we find that there is a lot of substance and material waiting to be discovered. Thus we have conducted this research to get a more in-depth understanding that we are all also deeply interested in.

1.3 The Aim of Our Project

As mentioned earlier, we wish to gain a greater and deeper understanding of how unit fractions work, and why it has such peculiar patterns. But apart from that, we also hope to educate not just ourselves, but also others like the judges or anyone who has an interest in this topic. Not to mention, learning this may or maynot come in handy for us in the near future in our normal academic mathematics.

1.4 A Preview of Our Research

Throughout the entirety of this project, we have been carefully handpicking and cultivating our questions, all of which are revolving around the idea of the splitting of fractions into unit fractions. There are 2 different aspects of how this process is broken down into, which are how this splitting can occur, and how this splitting occurs when we change certain properties of what we consider a unit fraction.

If you are unclear of what the splitting of fractions into unit fractions mean, here are some examples:

~ $\frac{1}{1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$

~ $\frac{1}{3} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1}$

2. Our Research Questions

In this research we have 3 big Research Questions,of which we will list here:

Q1. Can any given unit fractions be split into other distinct unit fractions?

Q2. Can any positive rational number less than or equal to 1 be expressed as a sum of distinct unit fractions?

Q3. Can any positive rational number more than 1 be expressed as a sum of distinct unit fractions?

Research Questions 1, 2 and 3 will be allocated in subtopics 3, 4, 5 respectively.

3. Can any given unit fractions be split into other distinct unit fractions? If so, how?

3.1 Defining RQ1

This question may or may not seem complicated, but nonetheless we will try to explain what it means. With this RQ, we are looking at whetheror not any unit fraction (which examples we have provided) can be split into other unit fractions. These "secondary unit fractions" we call it, must be different from each other and thus no duplicates would be entertained. To visually understand this, we have so kindly provided more examples.

$\frac{1}{1} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$, WRONG. Though this is technically correct, this is not our desired result as we want **distinct** unit fractions and there are 3 duplicates of 1/3.

$\frac{1}{1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$, CORRECT. This is what we want as the 3 secondary unit fractions are distinct.

We plan to find if ANY unit fractions can be split like this, and if so why? But if proven false, we will likewise find out why.

3.2 Our Basic Knowledge and Understanding of RQ1

Based on the question, we can find some clear-cut facts, one of which being: for a unit fraction to be split into secondary unit fractions, the secondary unit fractions must obviously be smaller than the original.

We can also tell that as you first minus off the first secondary unit fraction (SUF) from the original unit fraction (OUF), the denominators of the SUFs would be bigger. And as you keep minusing fractions and fractions, the SUF denominators will get larger and larger. To properly show this we will give a very simple algebraic expression here.

$\frac{1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \ldots$, where a < b < c. This is of course assuming that everytime we take out a SUF, it is the largest possible fraction where it is smaller than 1/n.

3.3 The Solution to RQ1

First off, from the OUF, $\frac{1}{n}$, we must first find the largest UF that is smaller than $\frac{1}{n}$. However, here's the easy part, since the numerator is 1 throughout, that means to find the largest SUF, we just have to take $\frac{1}{n+1}$. This works since any UF will be made smaller by the smallest amount by adjusting the denominator to be 1 more, as shown in this simple proof.

$\frac{1}{n} > \frac{1}{n+1}$. Now we form a common denominatorto get,

$\frac{n+1}{n(n+1)} > \frac{n}{n(n+1)}$, which simplifies to,

n+1 > n

Now, after we have removed 1/n+1 from 1/n, what we have left is 1/n(n+1). This can be concluded from the proof we have below.

Assume that "what we have left is $\frac{1}{n(n+1)}$" is true, which means

$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$. Likewise, we form acommon denominator and we get,

$\frac{n+1}{n(n+1)} = \frac{n}{n(n+1)} + \frac{1}{n(n+1)}$, which is true!

3.4 RQ1 Conclusion

With the simple proof that we have provided above, the answer to our RQ1 is yes, with very satisfying results. Any UF can be split into 2 other distinct UFs with the equation:

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$$

And we have examples to show it.

$$\frac{1}{7} = \frac{1}{8} + \frac{1}{56}$$

$$\frac{1}{100} = \frac{1}{101} + \frac{1}{10100}$$

Therefore, the answer to RQ1 is yes.

4. Can any positive rational number less than or equal to 1 be expressed as a sum of distinct unit fractions?

4.1 Defining RQ2

Basically what this RQ is asking for is if given anypossible fraction less than or equal to 1, it can be split into the same way as RQ1, with all theSUFs being distinct. The only difference between this and RQ1 is the fraction that we are splitting. For RQ1, the original fraction is a unit fraction, but this is for any positive UF less than or equal to 1. We can give an example of what we want.

$\frac{4}{5} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}$(We are not really using any formula for this equation)

4.2 Our Solution to RQ2

For the fraction equivalent to 1, just take $\frac{1}{1}$. For fractions that are less than 1 and more than0, we will use a method commonly known as the greedy method. This method is basically an algorithm that takes the largest possible unit fraction from the original fraction, and keeps on doing so until we have nothing left to take from it. Here, we will prove it true.

First, we say that the fraction we have chosen (which is less than or equal to 1) is $\frac{a}{b}$. Since $\frac{a}{b}$ is less than 1 and more than 0, we can get this inequality.

$\frac{1}{n} \leq \frac{a}{b} \leq \frac{1}{n-1}$

This inequality means that a/b is within the range or equal to any one of these two UFs. Like maybe

$\frac{a}{b} = \frac{3}{5}$

$\frac{1}{3} \leq \frac{3}{5} \leq \frac{1}{2}$

If a/b is a UF, then there is no need to solve it. In the next few parts of the explanation we will assume that a/b is not a UF and $\frac{1}{n} < \frac{a}{b} < \frac{1}{n-1}$.

Now, according to the greedy algorithm, we take the largest fraction we can possibly take (1/n) from a/b, and we get this equation.

$$\frac{a}{b} - \frac{1}{n} = \frac{an-b}{bn}$$

Whenever we take the largest UF from the chosen fraction, we get $\frac{an-b}{bn}$, and we keep taking the largest UF from this remaining fraction, we want to see this fraction drop in value, and we do so by looking at the numerator. If we can say thatevery single time we minus the largest UF from the remaining fractions, we can then say that it will eventually drop down to 1, making the last remaining fraction a *unit* fraction, therefore solving the equation. So, from $\frac{a}{b} - \frac{1}{n} = \frac{an-b}{bn}$, try to prove that an-b < a.

To do this we take the second inequality of $\frac{1}{n} < \frac{a}{b} < \frac{1}{n-1}$(this formula can be found in the previous page).

$\frac{a}{b} \leq \frac{1}{n-1}$. Now when we find a common denominatorand cancel it out we get,

⇒ a(n-1) < b

⇒ an-a < b

⇒ an-b < a

This proves that an-b will keep dropping till it hits 1.

However there is another possibility unaccounted for, which is $\frac{1}{n} > \frac{an-b}{bn}$. This inequality must be true or this method will not work. Using proof of contradiction, we assume the opposite, taking

$$\frac{1}{n} < \frac{an-b}{bn}$$

⇒ $\frac{a}{b} > \frac{2}{n}$ (as derived from $\frac{a}{b} - \frac{1}{n} = \frac{an-b}{bn}$)

But $\frac{2}{n} > \frac{1}{n-1}$, forming a contradiction. As compared to the inequality $\frac{1}{n} < \frac{a}{b} < \frac{1}{n-1}$.

That covers up the last loophole and we can say any fraction less than or equal to 1 can be a sum of distinct unit fractions.

4.3 RQ2 Conclusion

In conclusion, for any fraction less than or equal to 1, it is possible to split it up into several distinct positive integers by using a method known as the greedy method. As mentioned, it is a simple algorithm that takes the largest UF from your chosen fraction, and continues to do so until it leaves no remainder. We have a few examples of how the greedy algorithm can take place.

$$\frac{7}{13} = \frac{1}{2} + \frac{1}{26}$$

$$\frac{4}{13} = \frac{1}{4} + \frac{1}{18} + \frac{1}{468}$$

$$\frac{57}{82} = \frac{1}{2} + \frac{1}{6} + \frac{1}{36} + \frac{1}{1476}$$

Therefore the answer to our RQ2 is also yes.

5. Can any positive rational number more than 1 be expressed as a sum of several distinct unit fractions?

5.1 Defining RQ3

RQ3 is approximately the same thing as RQ2, just that instead of any fraction less than or equal to 1, we are now investigating any other fraction more than 1. Likewise, the expression that we eventually end up with must consist only of distinct unit fractions, like the example we have provided below.

$$\frac{3}{2} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{20}$$

5.2 Our Solution to RQ3

For RQ3 we have basically split our solution into2 different parts. One of which is proof that $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is not bounded (which we will later explain the purpose), and then a second part that takes this proof and the greedy algorithm in RQ2 to come up with an answer for this question.

5.2.1 Proof that $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is not bounded

This proof that this set of numbers added together is not bounded will come in useful in the third part of our solution, where we can safely say that this set can reach any positive rational number, over the value of 1 that is.

So, let us first take the set of unit fractions from $\frac{1}{1}$ to $\frac{1}{\infty}$.

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

From here, we start to group these fractions up, using the formula 2^n for n is (the group number) - 1. To understand this we look at the sum shown below, where each underlined part of this set is one group.

1+ <u>1/2</u> + <u>1/3 + 1/4</u> + <u>1/5 + 1/6 + 1/7 + 1/8</u> + <u>1/9 + ... + 1/16</u> +...... (these fractions sadly cannot be shown as correctly formatted fractions, or else we won't be able to underline them)

From here we start to modify the denominators in these fractions, such that the denominators are all the same as the last fraction in each underlined set, forming

1+ 1/2 + 1/4 + 1/4 + 1/8 + 1/8 + 1/8 + 1/8 + 1/16 + ... + 1/16 +......

By doing this we reduce the individual and overall total value of this equation, but if we prove that this set, that is smaller in value compared to the original, is not bounded, the original and larger-in-value set would likewise not be bounded either. So if we add up all the fractions in each of the underlined groups, we get

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$$

This will carry on infinitely as long as the number of fractions in a group follow the formulae above, 2^n for n is (the group number) - 1.

5.2.2 Combination of Two Theorems for A Final Result

With the proof above, we can say that the set of unit fractions can reach up infinitely, summing up to any unit fraction given. However, usingthis orderly set of $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$, we will reach a stage where $\ldots + \frac{1}{n} < \frac{a}{b}$, and where $\ldots + \frac{1}{n} + \frac{1}{n+1} > \frac{a}{b}$, for if $\frac{a}{b}$ is our chosen fraction.

This is where the greedy method in 4.2comes into play, in which we had this equation: $\frac{1}{n} < \frac{a}{b} < \frac{1}{n-1}$, **where $\frac{a}{b}$ is now the original $\frac{a}{b}$ minus $\frac{1}{1} + \frac{1}{2} + \ldots \frac{1}{n}$ for simplicity purposes.** Using the greedy method, we can take out the largest possible fractions from this a/b, and we know that it will eventually take out all the fractions within.

The UFs used whilst carrying out the greedy method will not be duplicated in the set before, $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$, because the UFs derived from the greedy method, will only get smaller and smaller, never bigger, Or else the loophole that we found in 4.2, where $\frac{1}{n} < \frac{an-b}{bn}$would be true, but we have since proven it false.

5.3 RQ3 Conclusion

As such, for fractions over one, we first use the set of consecutive unit fractions ($\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$) till a point where $\frac{1}{n} < \frac{a}{b} < \frac{1}{n-1}$ **where $\frac{a}{b}$is now the original $\frac{a}{b}$ minus $\frac{1}{1} + \frac{1}{2} + ... \frac{1}{n}$for simplicity purposes.** and we use the greedy method from there, like this example,

$$\frac{59}{30} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{120}$$

Therefore the answer to RQ2 is also yes.

6. Summary and Conclusion

To summarise our entire project work, we have researched on 3 different Research Questions, which we will list here again:

Q1. Can any given unit fractions be split into other distinct unit fractions?

Q2. Can any positive rational number less than or equal to 1 be expressed as a sum of distinct unit fractions?

Q3. Can any positive rational number more than 1 be expressed as a sum of distinct unit fractions?

In each of them we have provided 1 to 2 formulas and proofs to help support our answer, and oddly enough, the answer to all 3 of our questions are yes.

That means, that basically any positive fraction that you chose, whether a UF, valued less than or more than 1, it will still find a way to be expressed as a sum of distinct positive unit fractions, disregarding the number of unit fractions required.

Thank you for spending time reading our report. Havea nice day!

7. Sources

1. http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fractions/egyptian.html Sourced to explain the greedy method.
2. https://www.quora.com/What-is-the-sum-of-the-series-1+-1-2-+-1-3-+-1-4-+-1-5-up-to-infinity-How-can-it-be-calculated Sourced to understand why 1/1 + 1/2 + .., is not bounded.

~ Mentored by Mr Zong Lixing ~