# Lights Out! Extension

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# Abstract

This project is about the mathematical game, Lights Out puzzle, where the player plays using an electronic device. He has to turn a grid from one colour to another by selecting squares. Selecting a square toggles its 4 adjacent squares and itself to change colour. There are only 2 colours (on and off) and these squares change back to the original colour when the selected square is toggled again.

This project aims to find a method to solve the puzzle in the least number of moves, and explore its variations. The variations are based on the original puzzle but with changes to make it more challenging and interesting. We will be using algebra and number theory to find a method for the puzzles.

Last year, we found that it is possible to use algebraic variables and expressions to represent each square. We let the top row of the grid be the unknowns, and then deduce downwards. By forming and solving the simultaneous equations, we solve the puzzle. After altering the method, we also managed to solve the variation Lights Out X by separating and transforming it into basic puzzles.

This year, we were able to dive deeper into these puzzles and were able to explore and solve more of the variations of the puzzle, namely Lights Out Three Colour Coding, Lights Out Triangles, Lights Out 3D and Lights Out Warp.

# Contents

# 1 Introduction

## 1.1 Description

The Lights Out puzzle was first invented by Tiger Electronics. It is a mathematical puzzle using a square grid. It is usually done on electronic devices such as laptops and mobile phones, although the original version had its own device. The puzzle is usually a 5 x 5 grid with squares all grey.

When you start the game, all the lights are on (grey), and you have to solve it by “turning off” all the lights. The puzzle starts with a grid, all grey (lit up). It can be represented in a matrix like this 3x3:

0 0 0
0 0 0
0 0 0

Where 0 represents lit and 1 represents unlit. How we “turn” the lights off is by selecting squares. The selected square, along with its 4 adjacent neighbours, will switch its state. When it is 0, or lit, it turns to 1, which is unlit, and vice versa. So if middle square of the 3x3is selected, it becomes:

0 1 0
1 1 1
0 1 0

Last year, we were able to formulate a method to calculate the solution to this puzzle, as well as a variation (Lights Out X) using modulo mathematics and algebra. This year, we will be exploring further into this field by solving other variations, the Lights Out Three Colour Coding and Lights Out Triangles.

## 1.2 Objectives

We want to:

- Experiment on the Lights Out puzzles

- Find a method to solve for the most efficient way to solve the Lights Out puzzle variations, no matter the size of the board.

Some questions we ask are:
- How do we know if a solution is the shortest?
- Can there be more than one solution?
- Is there a relationship between the normal puzzle and its variations?

# 2 Literature Review

## 2.1 Original Lights Out

There has been plenty of research done on the original Lights Out Puzzle, but *none* on its variations that we are exploring in this project.

Sutner (1989) had stated and proved that “going from all lights on to all lights off is always possible for *any* size square lattice”. Barile, Margherita(2002) had made observations and found the solutions of the puzzle up to 7 x 7, but we aim to find a method to solve all square puzzles. Mathematician Rafael Losada, who wrote an article regarding the puzzle for the SUMA magazine, had come up with a solution using linear algebra. However, the solution and explanation is very complicated and we were able to simplify the solution last year.

We were able to solve the original puzzle last year by setting the top row of squares as unknowns, algebraic variables, and then deducing and representing the rest of the squares using those variables, row by row. Once the last row is reached, we are able to form simultaneous equations using each square in the row, and by solving the equations, we solve the puzzle.

## 2.2 Lights Out Variations

Pengfei li (2014) has done research and found an algorithm to determine if a pattern of Lights Out 3 Colour Coding can be all switched off, and how to do so. However, it is not the optimal solution.

There has been very little research done on the variations yet, thus we plan to explore them and come up with a solution method. We also want to explore the similarities between them and the original puzzle, how they relate to each other, and formulate a solution by altering the original method.

# 3 Methodology

## 3.1 Original Puzzle

Last year, we used our own method, using algebra to solve the original lights out puzzle.

For this example, we shall use the 3x3 grid.

| | | |
|---|---|---|
| a | b | c |
| d | e | f |
| g | h | i |

For every unknown, its value must be 0,1 for the number of times it has been pressed. If it has been pressed, its value is 1, if not, its value is 0.

Since each light must be turned off in the end, they have to be toggled an odd number of times. The squares surrounding **a**, including **a**, determine the value of **a**. Because of this, the squares surrounding **a** must be selected an odd number of times, meaning that **a**+**b**+**d** must be an odd number, which can be expressed mathematically as **a**+**b**+**d**=1(mod 2)

Following this logic in order to ensure that all nine other squares are lit up as well, the surrounding unknowns around a square must add up to =1(mod2). Thus, we can continue forming equations in a similar fashion down the board, with all mathematical operations done in modulo 2 . In a 3x3 equation, the following equations must be fulfilled:

**a+b+c=1, a+b+e+c=1, b+c+f=1, d+a+e+g=1, e+b+d+f+h=1, f+c+e+i=1,g+d+h=1,h+i+g+e=1 and i+h+f=1**

As long as all these equations are fulfilled, the lights out puzzle will be solved.

However, if we use this method, it is very difficult for someone to be able to solve the values of the unknowns manually unless they use a computer to test for all values of the unknowns. Hence, we have to find a way to simplify these expressions,

To do this we set the first 3 squares as unknowns, a, b and c to represent the rest of the gridss. If each light must be turned off in the end, they have to be toggled an odd number of times. Therefore, **a+b+d** must be an odd number, or expressed as **a+b+d**=1(mod 2). Now, by making **d** the subject, we can represent it using only **a** and **b**. We continued this in a similar fashion down the board, all the squares, with all mathematical operations done in modulo 2.

We get:

| | | |
|---|---|---|
| a | b | c |
| a+b+1 | a+b+c+1 | b+c+1 |
| a+c+1 | 0 | a+c+1 |

There are 3 unknowns and 3 equations, which are **b+c**=1, **a+b+c**=0 and **a+b**=1. From **b+c**=1 and **a+b+c**=0, we can see that **a**=1. Looking at **a+b**=1, we now get **b**=0. Since **b+c**=1, **c**=1.

Thus the solution we get:

| | | |
|---|---|---|
| 1 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 1 |

This method works for all square or rectangular puzzles, and there might be more than one solution.

This year, we will show how this relates to other variants we explored and how the method can be altered.

## 3.2 Lights Out Three Colour Coding

Lights Out Three Colour coding fundamentally functions in the same way as the original puzzle, except that each square has 3 states instead of 2, and each time a square is toggled, its state increases by 1 and will return to 0 if a square with state 2 is toggled, much like the original puzzle. The aim of this variation is to get the matrix into this:

2 2 2
2 2 2
2 2 2

### 3.2.1 Observations

Notice that:

- This variation is essentially the same as the original, and the only difference is that a square needs to be toggled 2 times if you want to reach the third colour. This means that we could use mod 3 instead of mod 2 in our solution to solve the puzzle.
- In mod 3, negative numbers can be converted to positive by subtracting it from 3. For example, in mod 3, -2=1.
- Unlike the original puzzle, each square should be selected 2 times at most instead of 1.

### 3.2.2 Solution

Since the only difference from the original is the number of times we need to toggle before it repeats, we just need to change the algebraic equations from mod 2 to mod 3. For example, this 3x3 here is presented in mod 3 in order to turn all lights to the third colour:

| | | |
|---|---|---|
| a | b | c |
| 2-a-b | 2-a-b-c | 2-b-c |
| 1+a-b+c | 2-a-b-c | 1+a-b+c |

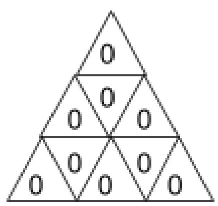
With the last row, we can now form three equations in mod 3:

2-**a**=2
-**b**=2
2-**c**=2

By solving them, we get that **a**=0, **b**=1, **c**=0. If we follow the algebraic expressions and select the square according to the value mod 3, it will toggle every square to the second colour.

## 3.3 Lights Out Triangles

Lights Out Triangles is a variation that changes the shape of the puzzle. Instead of using a square matrix, the puzzle utilises a triangular pyramid-shaped board. Each selected triangle still toggles itself and its adjacent neighbours, but the board looks as follows:



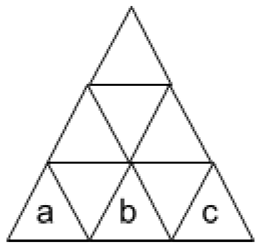

The aim of the game is still the same, to change the board from one colour to another.

### 3.3.1 Observations

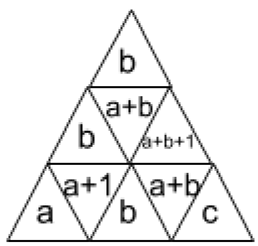
We can see how the triangular board has no definite top row like a square board does, so we cannot simply set the entire top row as unknowns. However, there is a clear bottom row where we can set our variables and work upwards.

### 3.3.2 Solution

We will start by setting the bottom row as algebraic variables as unknowns, before working upwards.




Using the same deduction method in modulo 2, we are able to obtain the following figure, slowly deducing towards the upper-right direction:




Now, by forming simultaneous equations (in mod 2) using the 3 expressions in the rightmost column, we get:

c+(a+b)=1
(a+b+1)+(a+b)+(a+b)=1
b+(a+b)=1

After solving, we get a=1, b=1 and c=1. By following the algebraic expressions and assigning values (either 0 or 1) to each small triangle, we are able to find a solution to the puzzle.

This method works on triangular boards of any size since we are always taking the rightmost column to create simultaneous equations, so the number of equations formed will always equal the number of variables, which is the bottom row.

## 3.4 Lights Out 3D

This variation, Lights Out 3D, involves using a 3-dimensional cube made of $n^3$ smaller cubes as the board instead of a flat square. The rules are the same, but the middle cube will now toggle 6 adjacent cubes, instead of the original 4 on a square board. It can be represented using N of N x N boards, with each board representing a layer.

## 3.4.1 Observations

We have observed that the 3D lights out cube has generally the same structure and solution as the original lights out puzzle. Since it has one more dimension than the original puzzle, we will be setting the first layer as algebraic variables instead of only the first row, as we had done in the original puzzle.

## 3.4.2 Solution

After we separate the cube into n layers, we solve it layer by layer. We first used a 3x3x3 cube, and split it into three 3x3 boards:

Layer 1

| | | |
|---|---|---|
| A | B | C |
| A+B+D+1 | A+B+C+E+1 | B+C+F+1 |
| C+G | B+H+1 | A+J |

Layer 2

| | | |
|---|---|---|
| D | E | F |
| A+D+E+G+1 | B+D+E+F+H+1 | C+E+F+J+1 |
| F+G+H | 0 | D+B+C |

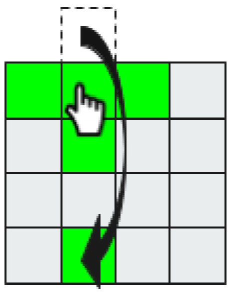
Layer 3

| | | |
|---|---|---|
| G | H | J |
| D+G+H+1 | E+G+H+J+1 | F+H+J+1 |
| A+J | B+H+1 | C+G |

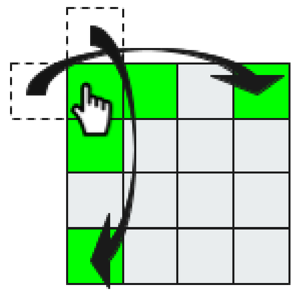
In each of the three layers, we first set the top 3 squares to be unknowns. We continue solving the 3x3x3 cube with the same method we used as for the square board. There will be 9 simultaneous equations since there are 9 unknowns, formed by using the last layer of the puzzle. This makes solving a single puzzle very tedious as there will be $n^2$ simultaneous equations for every puzzle with side length $n$.

## 3.5 Lights Out Warp

The Lights Out Warp puzzle is fundamentally different from the original puzzle, since the board can be seen as a loop with no starting and no finishing square. When a square on the edge is selected, not only are the neighbouring squares toggled, the directly opposite squares are toggled as well as if the square had “warped”.
For example, this figure shows the effects of selecting a square on the edge and corner of a 4x4 cube:



and



Our method of setting unknowns for the original puzzle will not be able to work on this since we are not able to deduce the rows with only one row of algebraic variables.

### 3.5.1 Observations

We realised that since every single square toggles 5 squares including itself, each square is also affected by 5 squares, including itself. This makes the question trivial since if we select **every square**, each square will toggle exactly 5 times and change state from lit to unlit, solving the puzzle.

Thus, the problem lies in finding out whether there are shorter methods to solve the puzzle and how to find them.

We are unable to deduce the puzzle with only the top row as algebraic variables because different from the original puzzle, the top row is affected by both the second and last row, instead of only the second.

However, this means that if we set both the first and last row as algebraic variables, we are able to deduce the second row and the rest of the grid.

### 3.5.2 Solution

We first set the first and last row as unknowns, then deduce the rest of the square and use the last two rows to form simultaneous equations.

This is the 4x4 grid formed:

| | | | |
|---|---|---|---|
| a | b | c | d |
| a+b+d +e+1 | a+b+ c+f+1 | b+c+d +g+1 | a+c+d +h+1 |
| e+f+h | e+f+g | f+g+h | e+g+h |
| e | f | g | h |

Now, by using the last two rows to form simultaneous equations, we get the following equations (simplified):
a+b+d+e=0
a+b+c+f=0
b+c+d+g=0
a+c+d+h=0
a=1
b=1
c=1
d=1

This shows that for the 4x4, the only solution is to select all the squares since all the algebraic expressions in the grid equals 1. This method works on all grids since the number of unknowns, $2n$, equals the number of equations which can be formed by the last two rows, also $2n$.

# 4 Conclusion

We have explored and solved the Lights Out 3 colour coding puzzle, the Lights Out Triangles puzzle, the Lights Out 3D puzzle and the Lights Out Warp puzzle, all by setting enough cells as algebraic variables, then deducing the rest of the board and forming simultaneous equations. The key to solving the puzzle is to set just enough unknowns needed to deduce the whole board, so as to reduce the number of simultaneous equations formed and making the process easier.

However, we were unable to find a mathematical proof to show our method uses the least number of moves and discover if the method can be shortened, since $n^2$ simultaneous equations are needed for a Lights Out 3D puzzle using the current method, making it very tedious.

# 5 References

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