Hex Roots

Written Report - Category 8 (Mathematics)
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1 Introduction

1.1 Description of Project

The project we have focused on for this year's project work is on the topic of Mathematics, category 8. We have decided to name our project "Hex Roots". This project revolves around a board game and the most effective strategies to use in that game in order to achieve a victory.

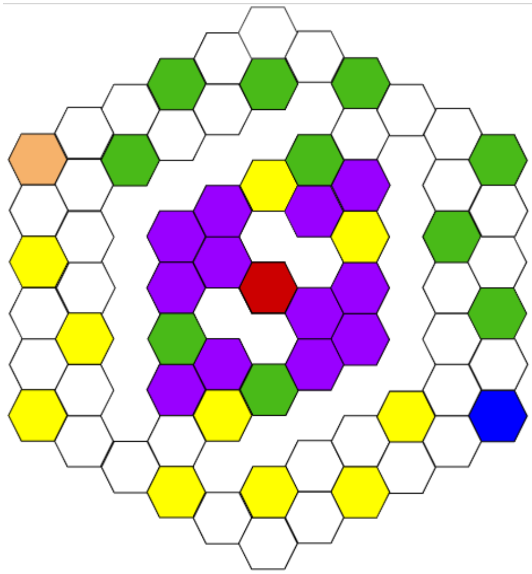
The board game is designed by us and would most accurately reflect the desired outcome of our research questions.

1.1.1 Terminology

Term	Definition
Board	The playing space in which the game takes place. Composed of 69 tiles of different colours with different functions for each tile.
Tile	An area on the board which is 1 hexagon large. A space where players can move onto. Different tiles have different abilities.
Move	A player's move for one turn.
Turn	A repeating sequence of 2 moves performed by the two players.
Adjacent Tiles	Two tiles sharing an edge.
In a Row	A subsequent action performed continuously, without cease, a number of times ($n > 1$).
Proximity	The number of moves necessary to get to a certain destination. Situationally affected by yellow or green tiles.

1.1.2 Game fundamentals

The board game is turn-based and involves a conflict between two players. The board is composed of 69 hexagonal tiles as shown below:



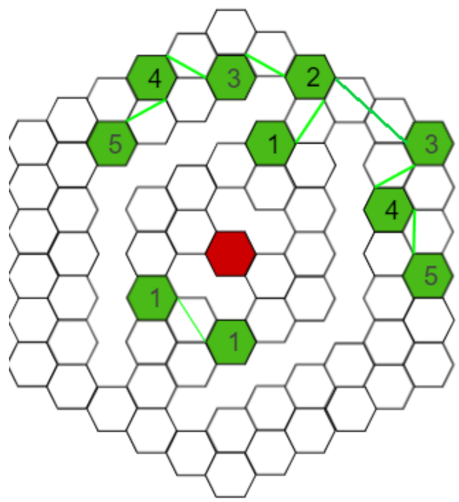
Different tiles are of different purposes and affect the game in different ways.

1.1.2.1 How to play the game

Two players, whose moves' orders are determined through random choice, start on the orange and blue tiles on the board (starting on blue or orange remains random as well). Every turn, each player gets a move; during a move, a player may move one step towards any direction and accordingly activate the function of the tile that they will step on (if any).

The below table shows the function of each tile.

Tile (colour)	Function
White	Allows the player to move in any direction to any adjacent tile.
Orange/Blue	Tile where players start on the first turn. The decision of which players start on which tile is made randomly. The players on the orange tile start first while the players on the blue tile start next.

Green	A special tile that upon landing on the tile, a player is allowed to jump onto the closest green tile in any direction. ("Closest" denotes proximity; the tiles are the closest as they need the least number of moves to travel between assuming only moving one tile at a time.) If there are green tiles with equal proximity, the player can choose which green tile to move to. Below is a visualisation of how a player can travel between green tiles.  (The numbers represent the proximity; tiles closest towards the red tile have the value 1. Green lines represent travel, from one green tile to another.)
Yellow	A special tile that upon landing on this tile, enables a player to "block" the other player. The other player will not be able to move to a tile of this player's choice for 3 turns. However, this can still be bypassed by using green tiles to jump over the blockage. Blocking any tile is allowed. An example is that if the red tile is blocked, it cannot be accessed by the other player, preventing winning for 3 turns. If a green tile is blocked and the other player steps on another green tile which leads to the blocked green tile, they cannot move to the blocked tile, forcing them the other way.
Purple	Similar to the white tile, except that if a player moves within

	the purple tiles for more than 3 turns in a row, they “die” and the other player wins.
Red	The player that lands on the red tile first wins. This is the most important tile and thus the players should make reaching it the highest priority.

1.1.2.2 Rules and Restrictions

Certain rules have been put in place to suggest solutions to difficult situations and to balance out the game, ensuring that no tile is too powerful and no player is disadvantaged or advantaged unfairly.

Rule 1. The players cannot move beyond the 69 hexagonal grids.

Rule 2. The players cannot be next to each other. If the player's way is blocked by the opponent, he is forced to move back to his previous tile.

Rule 3. The players cannot voluntarily forfeit their turn if they can move to another hexagonal grid.

Rule 4. However, if they have reached a limit on purple tiles, they cannot forfeit and instead lose.

Rule 5. If the destination of a green tile is blocked by the opponent, the player would be sent in the other direction when they move onto the starting green tile.

Rule 6. If the tile you are on becomes blocked, you are forced to move out of that tile within the next move.

Rule 7. The players cannot move to 2 or more yellow tiles within any 3 moves. (i.e. you must wait 2 more turns before being allowed to use a yellow tile.)

Rule 8. The players cannot move to 3 or more green tiles in a row. (This means that they can only travel once before having to “walk” again.)

1.2 Rationale, Objectives and Research Questions

1.2.1 Rationale

Our rationale suggests that through creating a game that is optimal for investigating game theory, we are able to explore the topic more and investigate more effectively the decision making process which is dependent on the mathematics involved in a situation. Through playing this game, we are able to connect the concept of a zero-sum game to decision making in value-based scenarios.

1.2.2 Objectives of research

Through the creation of Hex Roots, we are researching more about decision making and its relation to games where choices are made based on loss and gain. The game, based upon game theory, would let us more deeply understand the strategies to be undertaken to produce the best result in a competitive environment (that with the other player).

1.2.3 Research Questions

Research Question 1. What is the way to find the optimal strategy for winning the game with the utmost consistency and versatility?

Research Question 2. In what manner would decisions be made in this game and how would they involve both players?

Research Question 2a. At which stages of the game would there be conflict? How so?

Research Question 3. Could a strategy be made such that the opponent is impossible to win?

Research Question 3a. How much would the order of turns (i.e. who goes first) affect the outcome of the game?

1.3 Scope of Study

In this project, we are studying the effect of decision making and game theory, more specifically, zero-sum games where a player gains the other's loss.

2 Literature review

In this section, we have extracted information from various sources concerning game theory that is related to our project. To summarise from the different sources (Sources are at the bottom of this report):

Abstract. Our project concerns the topic of Zero-sum games. Zero-sum games are a certain scenario in game theory where there are two or more sides which play competitively. The defining aspect of this situation is that a player wins the other's loss. In total, the gain/loss of both players is equal to zero.

Article 1. Zero-sum games, though less common than non-zero-sum games, include many popular and well-known games such as Chess or tennis. A zero-sum game can have two or a million players, depending on the situation. As game theory plays a part in the financial market, the zero-sum game is also seen in the economy when it comes to options such as contracts, where if one investor loses, the money is transferred to the other. Non-zero-sum games on the other hand mean that the two players can work together to create a better outcome for both parties. In summary, Zero-sum games are the opposite of a win-win scenario and are strictly competitive.

We wanted to use this in our project as we knew that our game would be one of equal sum being zero. When a player performs an action, such as using a green or yellow tile, the opponent strays behind by a few tiles and the previously mentioned player becomes more ahead by the same amount of tiles. This is useful as it determines that if both players can cancel out an equal amount of tiles, the game is balanced.

Article 2. This article is about game theory and how it affects how games can be played. Using conclusive evidence, the article shows how some games can have a definite outcome if it is played perfectly and how other games can always result in one of the players winning if played perfectly. We used this article to find out if our game was one of perfect information (meaning that it is a constant-sum game) or it is a game of imperfect information (meaning that the game has a chance to lack a saddle point).

We realised that our game is one of perfect information. There is always a saddle point, and it is strictly determined. This gives us an idea of how we could construct strategies for this game. We would simply have to go for the moves that would give us the most benefit, or if there is a cost, we would choose another move that ensures that the benefit outweighs the cost. An example is in chokepoints. Say, if you were starting at orange and went towards the green side, you would most likely be choked by a smart player on the fourth green tile from your starting point. Thus, if this were anticipated, you would not instantly take the first green tile but take the second, losing 2 moves but gaining 3 at the same time. This still gives a +1 advantage, and making the opponent lose potentially a +3 for them (-3 for you), making you take advantage of 4 tiles' worth of moves, while the opponent gains no advantages (thus losing 4.) Opportunity is important in our game and thus preventing the opponent from having opportunities should be prioritised.

3 Study and Methodology

Through visual methodology and dissection of the board, we will be investigating if there is a way to find out the "best" strategy for this board game which is our primary point of research.

3.1 Methodology

We will be splitting the board into different segments. For both players, there are two routes that one can take - either the top or bottom path. We will be splitting the board into four sections - splitting the outer and inner layers of the board into top and bottom segments each to concoct a strategy.

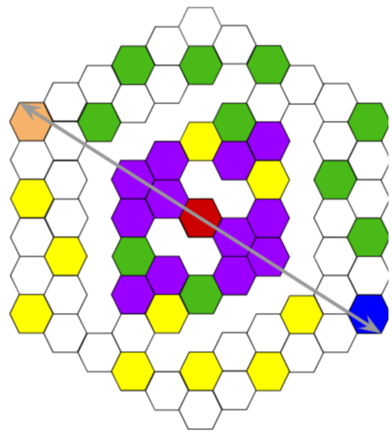
We will then propose an algorithm that will allow us to visualise a system of strategic identification, assigning values to tiles to have a more clear understanding of how the dynamics of the game tie in to the players' movements.

3.2 Splitting the Board

3.2.1 Top and Bottom

There are two primary ways that the board can be split up in this situation.
First, we will be splitting the board by one line as shown in the diagram. To simplify the ways to win, we must identify the advantages and disadvantages one can have by going through either route.

Visibly, going the top route would mean more access to green tiles and the bottom would mean yellow as the dominant tile, as shown below.



If you proceed to the top route, you have access to a faster means of travel. Green tiles will allow you to reach the red tile within less turns. However, if once a green tile is blocked, you will not be able to access it for one turn.

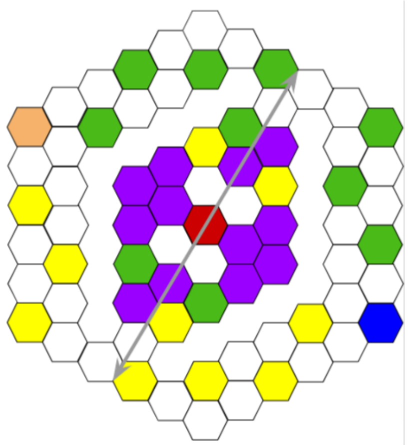
If you instead proceed through the bottom route, you will have more control over the other player. However, there is limited control as there is almost always an alternative route to get closer to the end (though sometimes slower). There is an advantage here, however, as the player can block functional tiles from the opponent (green or yellow tiles).
Sometimes, not using a yellow tile may be a more effective choice. For example, if you are leading, it is not a wise move to take a detour to block an opponent, as this may cause the opponent to catch up.

To sum up the usages of green and yellow tiles, it is best to fully utilise it only when it is absolutely necessary, and not using green or yellow tiles should not be treated as an advantage, but rather a tactical move, as the objective is just to reach the red tile. Using too many yellow or green tiles may lead to a disadvantage as the yellow tiles are at least 2 hexagons away from each other, and using all may bring you further away from the centre.

3.2.2 Left and Right

Secondly, we must consider the other more vertical splitting which divides the board into half.

This split concerns the two players, whose starting points are at opposite ends of the board. If it were proven that one player had an advantage against the other based on position, it can be said that the game is unfair. The main difference here is the arrangement of yellow and green tiles, as shown below.



The player who starts out on the left has more access to green tiles, though this does not affect the outcome as the other player still can get to the middle in the same number of moves.

Less access to yellow tiles, but since there is always a yellow tile within 4 moves, there is not much consideration for this.

If the player starts out on the right, the green tiles are the same except for one which has a 2 tile gap between one tile and another. Overall, however, the amount of move needed to reach the center is still the same.

Seemingly more access to yellow tiles, which is a pretty decent advantage. Note that it also offers a more strategic advantage in when or when not to use a yellow tile, as in the previous section, we have already established that yellow tiles need not be used every time. What is a big advantage for the player on the right, the 2nd player, is that he has more yellow tiles per hexagonal tile, which will give him more freedom to move and still be able to block the opponent.

This ultimately balances the game, as the 2nd player is able to win back the tiles he lost to the 1st player, by blocking the 1st player yet still moving towards the centre at a faster rate.

3.2.3 Beginning Decision

To come down to the choice of whether to choose green or yellow when starting out.

Green is the more advantageous choice for the player that starts first. This is due to the green tile that crosses a border, saving a total of 4 moves. Green tiles can help a player move at a faster rate, allowing the player to extend his lead over the other player, turning it into a blowout.

Yellow is also usable for the 1st player, as it can hinder the 2nd player's movement early on, but it is not as good as green, as the 2nd player easily has more yellow tiles than the 1st player and can easily turn the game around.

If you are starting out on blue, go for the yellow tiles. Notice how the yellow tiles are spaced out with 2 tiles in between, meaning that you would be able to use the tiles 4 times, and given the best scenario, it is able to increase the opponent's moves by 10 (considering 3 crucial tiles - one large-gap green tile, one chokepoint and one middleground chokepoint) and 1 green tile, as well as a chance at making the opponent move 3 times in the purple tiles, and thus victory.

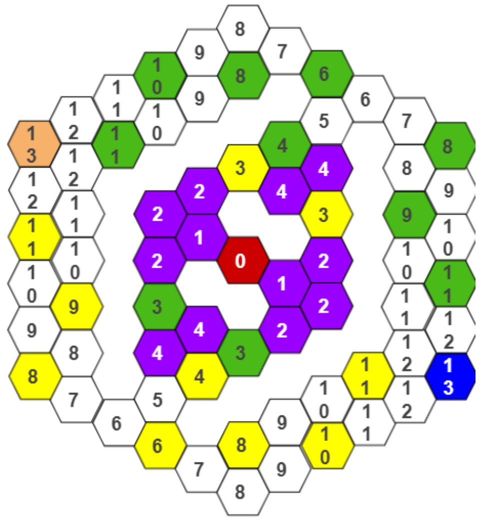
3.3 Visual Algorithm

Now that we have the necessary information, let us concoct an algorithm that will allow us to visualise a system of strategic identification. We will be doing this through a diagram.

The system we will be using is dependent on proximity. The algorithm will follow this pattern:

Each tile will be given a number to denote the proximity from the centre red tile. It starts with the red tile and branches out along the board until it reaches the blue and orange tiles. This can be achieved through making the next tile have a value being the value of the adjacent tiles + 1. Given multiple adjacent tiles of different values, the one with the lowest value will be considered.

The complete visualisation is shown below.



Here we have the numbered tiles with values.

Notice that it takes 13 moves to reach the centre (not factoring in green tiles). Now, we can apply the green and yellow tiles to the algorithm. Think of it as if green tiles were the only way to reach the red tile. In that case, it would only take 8 moves to reach the middle. Not going with the green tiles would mean an addition of value, thus missing a green tile would get the player +2 value because you would take 2 more moves. If a person is taking the yellow route, their objective would be to increase their opponent's moves so that they can get there first.

A player is either behind or in front of the other, meaning that the players' distance from each other are an equal sum, one being positive and the other being negative. Thus, it appears that this game is one of equal sum = 0.

A mathematical tally of total opportunities of the players, given that they will skip a potential tile if they are on cooldown unless the opponent can be blocked for as long or longer than the time they take to go back and use the tile (yellow tiles can hinder the opponent by 1-3 moves):

Green and Red represent Best and Worst case scenarios respectively.

Orange - Green:
$\min(0 + 2 + 2 + 1, 0 + 2 + 2 + 3) = 5$
$\max(0 + 2 + 2 + 1, 0 + 2 + 2 + 3) = 7$
Average → $(5 + 7) / 2 = 6$

Orange - Yellow:

min(0 + 1 + 1 + 1, 0 + 3 - 1 + 3 - 2 + 3 + 3 - 1 + 3) = 3
max(0 + 1 + 1 + 1, 0 + 3 - 1 + 3 - 2 + 3 + 3 - 1 + 3) = 11 (near-impossible best case scenario)
Average → (3 + 11) / 2 = 6

Blue - Green:
min(0 + 2 + 3 + 1, 0 + 2 + 3 + 3 - 3 + 3) = 6
max(0 + 2 + 3 + 1, 0 + 2 + 3 + 3 - 3 + 3) = 8
Average → (6 + 8) / 2 = 7

Blue - Yellow:
min(0 + 1 + 1 + 1, 0 + 3 - 1 + 3 - 1 + 3 - 1 + 3) = 3
max(0 + 1 + 1 + 1, 0 + 3 - 2 + 3 - 1 + 3 - 1 + 3 - 1 + 3) = 10 (near-impossible best case scenario)
Average → (3 + 10) / 2 = 6.5

We can show our results with this following table:

Data averages

<table>
<tr><td colspan="2"></td><td>6 (Blue-green)</td><td>8 (Blue-green)</td><td>3 (Blue-yellow)</td><td>10 (Blue-yellow)</td><td></td></tr>
<tr><td colspan="2"></td><td colspan="2">7 (Blue-green)</td><td colspan="2">6.5 (Blue-yellow)</td><td>6.75 (Blue)</td></tr>
<tr><td>5 (Orange-green)</td><td rowspan="2">6 (Orange-green)</td><td colspan="2" rowspan="2">6.5</td><td colspan="2" rowspan="2"></td><td></td></tr>
<tr><td>7 (Orange-green)</td><td></td></tr>
<tr><td>3 (Orange-yellow)</td><td rowspan="2">6 (Orange-yellow)</td><td colspan="2" rowspan="2"></td><td colspan="2" rowspan="2">6.25</td><td></td></tr>
<tr><td>11 (Orange-yellow)</td><td></td></tr>
<tr><td></td><td>6 (Orange)</td><td colspan="5"></td></tr>
</table>

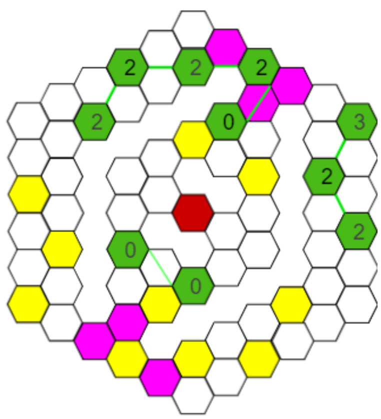
*Note that the average for Blue is 6.75 and the average for Orange is 6.

This supports our explanation that Green is more advantageous than yellow. However, if yellow is utilised properly, it can impact the game by a lot.

At first glance, it may seem that Blue is superior to Orange. However, note that the deviation caused by yellow tiles is less in Orange compared to Blue, as orange has a more standard arrangement while blue has a more sporadic but more abundant arrangement of yellow tiles. Thus, there is a higher chance for usage of yellow tiles on the Orange side.

3.3.1 The Art of Blocking

Let's give each green tile a "value" which is the number of moves it saves. When a player uses these tiles, their number of moves left gets subtracted by the value. Some green tiles do not have the value as they do not save moves.



Above: Highlighted chokepoints, green tiles assigned values and yellow tiles.

For example, you can block at the chokepoints (highlighted in pink), you can block green tiles effectively rendering them useless due to no green tile being able to save >3 moves, and you can also block other yellow tiles. You can even block the red tile if you like, which could cause your opponent's defeat (however, both players have access to yellow tiles in the middle of the map.)

Blocking a green tile increases the enemy's moves by 2-3. Blocking a chokepoint increases the enemy's moves by 3 if there is no way in. Blocking a yellow tile does not increase the enemy's moves but instead cuts off the opportunities for enemy blocking. Typically, this would save you 3 moves if no other yellow tiles are within reach of the opponent. A critical spot to block is in the centre where few yellow tiles are available. If one decides to go through the yellow route, blocking the important tiles would be of high priority.

For the green tile route, the player should carefully decide whether to proceed or to choose not to use the green route such that the opponent can be tricked into thinking that they would be gaining moves, but in reality they would only be gaining 1 or 2 moves which presents trouble to the opponent. Once approaching the middleground, block the opponent's middle ground tile immediately and rush towards the red tile while ensuring that the opponent has no near access to yellow tiles. This should ensure a victory.

4 Conclusion

4.1 Outcomes, Analysis & Discussions

4.1.1 Research outcomes

In this section, we conclude the project and answer the research questions we had proposed at the beginning.

1 What is the way to find the optimal strategy for winning the game with the utmost consistency and versatility?

Through our use of values which apply to different situations to decide the best possible course of action which is to be taken, we have discovered that it really depends on the situation - different possibilities will lead to different actions to be carried out. However, we were able to use the system to conclude about what is the best action in various general scenarios - what route to take given the opponent's input and how to most effectively progress through different parts of the board.

Using this method we also factored in the different aspects of the board - purple, yellow, green tiles etc - and explained their usage in strategies. In game theory, the topic we are researching about, this game is an example of a zero-sum game where a player is either ahead or behind another player. Calculating the gain/loss based off of a certain action expressed in tiles would thus be the base of strategy deduction.

2 In what manner would decisions be made in this game and how would they involve both players?

Decisions involve the gain/loss of tiles towards the centre tile, as said before. The decisions revolve around the usage of green/yellow tiles, methods to hinder the opponent's movement such as blocking important tiles, or taking different routes to maximise your strategy's efficiency.
While many decisions directly affect two players, some decisions are used to benefit a player specifically, like the usage of green tiles, the tactics that are used in order to avoid being a victim of the other player and the decisions that are made to bypass the purple tiles.

2a At which stages of the game would there be conflict? How so?

Conflict would occur where opportunities are abundant for both players.

This includes at critical tiles where an action could prevent the other player from advancing for a few turns, giving a major advantage; this also includes in the middle, where the players are close to winning or losing.

In the middle, it is a very high-stakes situation as there is access to both green and yellow tiles as well as purple tiles which may contribute to the win of a player.

At the sides, if both players choose to take the same route, there might also be conflict, although this will be rare as in both left and right sides, one player has a slight disadvantage.

For example, on the top, the 1st player has a large advantage over the 2nd player if both use the same route, as he can get to the green tiles quicker and pick up a huge lead. Meanwhile, the 2nd player has an advantage in the bottom, such that they have more variety of yellow tiles at different positions tiles to choose from which makes his path to the centre easier.

3 Could a strategy be made such that the opponent is impossible to win?

As the game is based on equal sum of both the player's advantage/disadvantage (which equates to zero at all times), the opponent always has a possibility of winning if they anticipate said strategy.

In theory, the opponent could also utilise the same exact strategy for a 50% chance of winning. It also matters which turn you take, first or second - some strategies may benefit from being first, vice versa.

Hypothetically, if the players were to think to an infinite extent and anticipate each other's moves an endless amount of times, they would be stuck in the game and would not know what to do or simply not move due to being unable to know what the other player is going to do, which in turn does not know what this player is going to do. This is a contradiction against the hypothetical "best strategy" and thus no strategy can have a 100% success rate.

However, using probability, one can rely on the most successful strategy and use that for a higher chance of winning.

3a How much would the order of turns (i.e. who goes first) affect the outcome of the game?

As previously mentioned, some strategies benefit from going first (like ones which utilise green tiles for speed) while some strategies would benefit more from the player going second in turn (strategies which go the yellow route. There would be more chances to anticipate the opponent's actions and block accordingly.)

While going first might give one a head start, it must also be considered that a strategy, by going second, is more planned and thus has more control over the other player. Sometimes the second player is more coordinated and has opportunities to make better use of the tiles as the first player's moves are easily anticipated, which gives the first player a reason to think of alternative movements or different approaches in their strategy to avoid being "trapped" by the opponent.

If the first player is capable, they are able to quickly enter the central area and also block the opponent from entering the area. This should be able to ensure high chances of a win.

4.2 Implications and Recommendations

4.2.1 Implications

We think that there are quite a few implications within this game.

Firstly, one implication is the value of defense over offense in our game. For example, in our game, we have designed many defensive rules that allow the defending player to have a chance against the offensive player.

For example, one such rule is the green tiles rule. It prevents players who use the green tiles to quickly finish the game and allow the other player to have a fighting chance. With the offensive player not allowed to move to a 3rd straight green tile, the defensive player who cannot move as fast can use the yellow tiles to slow the green tile user.

Unlike many offensive oriented board games, which prioritizes offense over defense, our board game is more defensive oriented. This has huge implications as an offensive minded player cannot succeed in this game, and this forces players to use methods not commonly seen in other games such as playing defense.

This changes the player's train of thoughts when playing this game, and will help the player train to think differently, which will definitely be beneficial in real life scenarios where resources are limited and being conservative is simply the better choice than compared to being reckless, just as being a defensive oriented player is a better choice than being an offensive one.

4.2.2 Recommendations

We feel that we have to recommend the player who will be trying our board game to think differently, as compared to other zero-sum games that are primarily focused on offense.

In fact, as our game is drastically different from other board games, there will definitely be a steep learning curve, but the rewards are worth it, as this different experience will allow the players to learn some unorthodox strategies to help them succeed, which will surely benefit them in life later.

All in all, this game helps to widen the area of the player's thinking, where they have to use unorthodox and new strategies in order to succeed, causing this board game to be an enriching new insight for players and us.

5 Sources Used

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