Hwa Chong Institution

Project Report

The Moon and the Sixpence

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August 2020

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1 Abstract

Our project The Moon and the Sixpence: Mathematics behind Coins is a study about using number theory and algebra approaches to calculate the Frobenius numbers of the sets with certain properties and hence determine whether there exists any winning strategy for the game Sylver coinage.

Some of our objectives are:

- To find appropriate methods to compute the Frobenius numbers of the sets with certain properties.
- To find whether a winning strategy exists for the players in the game the Sylver coinage.
- To determine if there are some hidden properties in the denominations that are chosen in a winning strategy.

Through our research and study conducted, we can provide an algorithm to calculate the Frobenius number for the Frobenius Coin problem as well as some winning strategies for the game Sylver Coinage. We hope that our research can help gain some insight related to the structure of both mathematical problems and hence can help develop some meaningful ideas for future work.

2 Introduction

2.1 Brief Introduction

In our project The Moon and the Sixpence: Mathematics behind Coins, we aim to explore on two problems known as the Frobenius coin problem and the Sylver coinage using number theory and algebra approaches.

2.1.1 Frobenius Coin Problem

The Frobenius coin problem, named after the German mathematician Ferdinand Frobenius is a mathematical problem that asks for the largest monetary amount that cannot be obtained using only coins of specified denominations. The solution to this problem for a given set of coin denominations is called the Frobenius number of the set.

For example, the largest amount of money that cannot be obtained using only coins of 3 and 5 units is 7 units. Any amount of money that is greater than 7 units can be obtained (e.g. $8 = 3 + 5, 9 = 3 + 3 + 3, 10 = 5 + 5$) Thus, the Frobenius number of the set [3,5] is 7.

2.1.2 Sylver Coinage

Sylver coinage is a mathematical game for two players, invented by Princeton mathematician John H. Conway. The game is named after the British mathematician James Joseph Sylvester, who proved that if a and b are relatively prime positive integers, then $(a-1)(b-1)-1$ is the largest number that is not a sum of non-negative multiples of a and b.

The following are the rules of the game: two players take turns naming positive integers greater than 1 that are not the sum of non-negative multiples of previously named integers. The player who cannot name such an integer loses.

2.2 Terminology

Following the convention of studies about Sylver coinage, we accept the following set of notations in our paper for the sake for convenience and conciseness.

- ***Positions*** are states of the game after one player has made the move.
- $\boldsymbol{N}$-position is a position that the player next to play can win. $\boldsymbol{P}$-position is a position that the previous player can win.
- Position $\boldsymbol{S}[x_1, ...x_j]$ denotes the position with previous moves $x_1, ...x_j$.
- We denote position $S'[x_1, ...x_j, a]$ as Sa (adding move a to position S).

- The largest available move at position S is denoted by $m(S)$. When $m(S)$ does not exist, we denote by $m(S) = \infty$.
- Symmetric game refers to the games in which either player has the same set of available moves with the same results.

2.3 Research Problems

- Are there some sets of numbers where there exists an efficient way of calculating the Frobenius number of?
- Does there exist a winning strategy for the game Sylver coinage?
- Does there exist some hidden properties in the denominations that are chosen in a winning strategy?

2.4 Objectives

- To find appropriate methods to compute the Frobenius numbers of the sets with certain properties.
- To find whether a winning strategy exists for the players in the game the Sylver coinage.
- To determine if there are some hidden properties in the denominations that are chosen in a winning strategy.

2.5 Rationale

We have noticed that both Frobenius Coin problem and Sylver coinage attract the attention of many mathematicians. However, for the Frobenius Coin problem, only one theorem is found by mathematicians, which is only applicable to a set of two relatively prime numbers, and a general formula is yet to be found. For the Sylver Coinage game, there also only exists one winning strategy, which merely states what number should the first player choose as an opening number whilst no detailed guidelines are provided after that.

As such, we are interested in finding possible methods to calculate the Frobenius numbers of sets of numbers with other special properties, and to come up with new algorithms for finding the Frobenius number. Furthermore, we aim to investigate different positions of the Sylver Coinage game and to determine if specific positions are winning or losing.

2.6 Fields of Mathematics

- Number Theory
- Game Theory
- Algebra

3 Methodology

- Read up on articles of previous research involving the Frobenius coin problem and Sylver coinage to understand the mathematical mechanism behind these two problems.
- Find specific sets of numbers with certain properties for which calculating the Frobenius number is possible.
- Determine the appropriate methods to calculate the Frobenius number of those sets of numbers
- Research and experiment with different numbers to try to find appropriate methods to win the game Sylver coinage.
- Determine the general properties in the denominations that are chosen of the winning strategies for Sylver coinage.

4 Literature Review

4.1 Frobenius Coin Problem

The Frobenius coin problem has been studied and discussed by many mathematicians, and a well-known theorem for certain cases of this problem is called the Chicken McNugget theorem.

Theorem 4.1 (Chicken McNugget theorem)**.** *The Chicken McNugget theorem, which is in the context of purchasing chicken nuggets in McDonald's, states that for any two relatively prime integers m and n, the largest integer that cannot be expressed in the form $am+bn$ for non-negative integers a and b is $mn-m-n$.*

Proof. We call an integer expressible if it satisies the condition in the question.

Consider the set $A = \{1, 2, ..., m-1\}$. Note that for any c coprime to m, $c \cdot A = \{c, 2c, ..., (m-1)c\} \equiv A \pmod{m}$.

Lemma: For any non-negative integer $c < m$, cn is the least expressible number $\equiv cn \bmod m$.

Proof: Suppose on the other hand that $cn - dm$ is expressible for some $d > 0$, i.e. $cn - dm = am + bn$ for some non-negative integers a, b. Thus, $(a + d)m = (c - b)n$, which implies that $(a + d)$ is a multiple of n (since $\gcd(m, n) = 1$). Let $(a + d) = gn$ for some positive integer g, and substitute to get $gmn = (c - b)n$. Because $c < m$, $(c - b)n < mn$, and $gmn < mn$. We divide by mn to get $g < 1$. Therefore, we have a contradiction, and cn is the least expressible number congruent to $cn \bmod m$. □

This means that because cn is expressible, every number that is greater than cn and congruent to it $\bmod m$ is also expressible. Furthermore, $cn-m$ is the greatest number $\equiv cn \bmod m$ that is not expressible. $c \leq m - 1$, so $cn - m \leq (m - 1)n - m = mn - m - n$, which shows that $mn - m - n$ is the greatest number in the form $cn - m$. Any number greater than this and congruent to some $cn \bmod m$ is expressible, because that number is greater than cn. All numbers are congruent to some cn, and thus all numbers greater than $mn - m - n$ are expressible. □

Corollary: For any integer k, exactly one of the integers $(k, mn - m - n - k)$ is not expressible.

Proof. We can set $k \equiv cn \pmod{m}$ for some $0 \leq c \leq m - 1$.
Case 1: $k \leq cn - m$. $mn - m - n - k \geq mn - m - n - (cn - m) = n(m - 1 - c) \geq 0$. Since $n(m-1-c)$ is the least number congruent to itself $\pmod{m}$ that is expressible, $k < cn - m$ is not expressible; $mn - m - n - k \geq n(m - 1 - c)$ is expressible.

Case 2: $k > cn - m$ $mn - m - n - k < n(m - 1 - c)$. Following the same argument, $k > cn - m$ is expressible; $mn - n - m - k < n(m - 1 - c)$ is not expressible. □

4.2 Sylver Coinage

Unlike many similar mathematical games, the Sylver coinage game has not been completely solved due to most games having an infinite number of possible moves to consider. However, there is a theorem by R.L. Hutchings which states that any of the prime numbers except 2 and 3 (5,7,11,13, etc.) wins as the first move although very little is known about the strategy after that. Furthermore, no one currently knows whether there are any winning opening moves at all that are not prime.

5 Working Progress

5.1 First Stage

Are there some sets of numbers where there exists an efficient way of calculating the Frobenius number of?
In this stage of our research, we have investigated on the algorithms of calculating the Frobenius number of a given set. Although an algorithm for calculation of Frobenius number of any given set with low time complexity is unlikely to be found, we have explored on some inventive general algorithms, as well as improved algorithms for calculation of Frobenius number of special set of integers.

Theorem 5.1 (Expansion Algorithm)**.** *The number of non-negative integer solution* $[x_1, x_2, ..., x_n]$ *of equation* $a_1x_1 + a_2x_2 + ... + a_nx_n = k$ *is equal to the coefficient of the term* y^k *of the formal power series expansion of the following rational function at the origin.*

$$F(y) = \frac{1}{(1 - y^{a_1})(1 - y^{a_2})...(1 - y^{a_n})}$$

Proof. Suppose $0 < y < 1$.
Consider the infinite series:

$$1 + y^{a_i} + y^{2a_i} + ... = \sum_{k=0}^{\infty} y^k a_i$$

By summation of geometric progression, the sum of this infinite series is equal to $\frac{1}{1-y^{a_i}}$.
Hence, the expansion of the function $F(y)$ is equal to:

$$(1 + y^{a_1} + y^{2a_1} + ...)(1 + y^{a_2} + y^{2a_2} + ...)...(1 + y^{a_n} + y^{2a_n} + ...) = \prod_{i=1}^{n}(\sum_{k=0}^{\infty} y^{ka_i})$$

Finally, consider the term $c_k y^k$ in the final result of summation. Every y^k is given by a unique path of summation and correspond to one solution $[x_1, x_2, ..., x_n]$. Hence the coefficient c_k is equal to the number of non-negative solutions. □

This result provides us with a general way of calculating the Frobenius number for a group of integers directly.

The Frobenius number for the set of integers, $[a_1, a_2, ..., a_n]$ can be defined as the largest integer k that produce no non-negative integer solution to the equation $a_1x_1+a_2x_2+...+a_nx_n = k$. Hence, by writing out the unique power series expansion of the function $F(y)$, the power k of last term with a zero coefficient is the Frobenius number of the set.

Example:

Find the Frobenius number k of the set $[4, 6, 7]$.

$$F(y) = \frac{1}{(1-y^4)(1-y^6)(1-y^7)}$$

Since the Frobenius number for the subset $[4, 7]$ is $4 \times 7 - 4 - 7 = 17$ by Chicken Macnugget Theorem, $k \leq 17$. The first 17 terms of Maclaurin expansion of $F(y)$ is given by:

$$F(y) = 1 + y^4 + y^6 + y^7 + y^{10} + y^{11} + 2y^{12} + y^{13} + 2y^{14} + y^{15} + 2y^{16} + y^{17} + ...$$

The last term with coefficient=0 is y^9.
Hence, the Frobenius number of the set $[4, 6, 7]$ is 9.

Remarks. The Expansion Algorithm provides a direct approach to calculate the Frobenius number of any set of integers. For a large number of big integers, since the only information it requires is the value of $1, 2, ...$ order derivatives at origin of a given rational function. As the accurate value of the derivatives are not necessary (we only need to determine whether it is 0), the process is not too troublesome by running a computer program.

However, it is obviously too troublesome for calculation of Frobenius number of a small set of small integer numbers. Algorithms with much less complexity should be found to solve the Frobenius problem for smaller or special set of integers.

Theorem 5.2. *If* $\gcd(a,b,c) = 1$ *and* $c|lcm(a,b)$*, then* $G(a,b,c) = lcm(a,c) + lcm(b,c) - a - b - c$ *where* $G(a,b,c)$ *is the Frobenius number for a set of numbers a,b,c.*

Proof. Define the modified Frobenius number $F(a,b,c)$ as $G(a,b,c) + a + b + c$.
It is obvious that $F(da,db,dc) = dF(a,b,c)$. Since $\gcd(a,b,c) = 1$ and $c|\text{lcm}(a,b)$, set $a = pr$, $b = qs$, $c = rs$ where $p,q,r,s \in \mathbb{Z}$

Note that $F(a,b,c) = r[F(p,qs,s)] = rs[F(p,q,1)] \implies F(p,q,1) = G(p,q,1) + p + q + 1$

As defined, the Frobenius number of p,q,1 is the largest number that cannot be obtained from $px_1 + qx_2 + x_3$. Since x_1, x_2 and x_3 are all non-negative integers, all numbers from 0 to infinity will be able to be obtained.

Hence, $G(p,q,1) = -1$ which leads to $F(p,q,1) = G(p,q,1)+p+q+1 = -1+p+q+1 = p+q$.

$$F(a,b,c) = rs[F(p,q,1)] = rs(p+q) = prs + qrs$$

Note that $\text{lcm}(a,c) = prs$ and $\text{lcm}(b,c) = qrs$. Thus $F(a,b,c) = \text{lcm}(a,c) + \text{lcm}(b,c)$.Finally, we obtain our result $G(a,b,c) = F(a,b,c) - a - b - c = \text{lcm}(a,c) + \text{lcm}(b,c) - a - b - c$. □

This allows us to quickly calculate the Frobenius number for a set of 3 numbers $\{a,b,c\}$ when $gcd(a,b,c) = 1$ and $c|\text{lcm}(a,b)$. For example, for $\{6,10,15\}$, since $gcd(6,10,15) = 1$ and $6|\text{lcm}(a,b)$, $G(a,b,c) = \text{lcm}(10,6) + \text{lcm}(15,6) - 10 - 15 - 6 = 29$.

Remarks. Note that if a,b,c are in the form xy, yz, zx, then **Theorem 5.2** states that the Frobenius number is $2abc - ab - ac - bc$. We can generalise this to **Theorem 5.3** below.

Note that if a set of positive integers, contain two which are coprime, say m, n, then it is easy to see that the Frobenius number is $\leq mn - m - n$, by simply considering those two numbers. Exactly what it is can be easily found by working backwards from the number.

However, what happens when there are no two coprime integers? Our following theorem will help.

Theorem 5.3. *Let n integers $a_1, a_2, \ldots, a_n$ be pairwise relatively prime. Their Frobenius number is*

$$a_1 a_2 \ldots a_n \left(n - 1 - \sum_{k=1}^{n} \frac{1}{a_k} \right)$$

Proof. First, we show that that number cannot be obtained. Suppose $(n-1)a_1a_2...a_n - a_1...a_{n-1} - a_1...a_{n-2}a_n - ... - a_2a_3...a_n = x_1a_1...a_{n-1} + x_2a_1...a_{n-2}a_n + x_na_1...a_{n-1}$.

Then, taking $\pmod{a_1}, a_1 | x_n + 1$. Similarly, $a_i | x_{n+1-i} + 1$.

Hence, $x_i \geq a_{n+1-i} - 1$. Thus, $(x_i + 1)a_1...a_{i-1}a_{i+1}...a_n \geq a_1...a_n$.
Then, we must have $(n-1)a_1...a_n \geq na_1...a_n$ which is a contradiction.

Now, we show that any number more than that can be obtained.

We use induction.
The case $n = 2$ is done by the Chicken McNugget theorem.

Take any n. Let $X = a_2a_3...a_n$
Note that $0, X, 2X, ..., (a_1 - 1)X$ is a complete residue set $\pmod{a_1}$.

Thus, for any $N > a_1a_2 \ldots a_n(n - 1 - \sum_{k=1}^{n} \frac{1}{a_k})$, we can take $cX \equiv N \pmod{a_1}$ with $0 \leq c < a_1$ But

$$N - cX > (n-1)a_1a_2...a_n - a_1...a_{n-1} - a_1...a_{n-2}a_n + ... - a_2a_3...a_n - (a_1 - 1)X$$

$$a_1((n-2)a_1a_2...a_{n-1} - a_1...a_{n-2} - a_1...a_{n-3}a_{n-1} + ... + a_2a_3...a_{n-1})$$

which by the induction hypothesis can be obtained.

This concludes our proof.

□

5.2 Second Stage

1. To find whether a winning strategy exists for the players in the game the Sylver coinage.
2. To determine if there are some hidden properties in the denominations that are chosen in a winning strategy.

In this stage of our research, we will investigate on the most popular problem in this game: what are the winning opening moves for the first player?

We will first re-cover the classic result by R.L. Hutchings, that any prime number larger than 3 wins as the first move using elementary Combinatorics tools.
Secondly, we will investigate on an explicit winning strategy in special positions based on our finding in the first stage.

Theorem 5.4 (Strategy Stealing Argument). *The second (previous) player cannot have a guaranteed winning strategy in any symmetric game, with the number of available moves* ≥ 3.

Proof. Suppose a guaranteed winning strategy exists for the second player in a symmetric game. By picking an arbitrary move, in the new position of the game, the first player now become the second player and can use the second player's "winning strategy", which is a contradiction. □

This theorem has provided insights for us to construct an $\boldsymbol{N}$-position.

Theorem 5.5 ($\boldsymbol{N}$-position 1). *Any position* S *with only one move that is available and does not eliminate any other move is a* $\boldsymbol{N}$*- position, and that move must be* $m(S)$*, which can be eliminated by picking any other move.*

Proof. By lemma 1, the largest available move $m(S)$ does not eliminate any smaller available moves. Hence, the move must be $m(S)$ if only one move that is available and does not eliminate any other move exists in the position.

Consequently, any other move (all other available moves are smaller than $m(S)$ will eliminate $m(S)$. This is proven by a finite ascend process. Choose any available move $a_0 \leq m(S)$. It necessarily eliminate at least one other move, and all moves it eliminates must be larger than it. We name the largest move it eliminates a_1. This move necessarily eliminates at least one other move larger than it. By picking the largest move that $a_0, a_1, a_2 ...$, the sequence will finally arrive on $m(S)$.

Consequently, the next player can always obtain a winning strategy by choosing play or not play move $m(S)$. Since the positions $Sm(S) = Sa_0m(S)$ for any step a_0, the next player can always steal the winning strategy of the previous player, if exists, by strategy stealing argument. □

This explicit construction of $\boldsymbol{N}$-position directly leads to a guaranteed winning strategy of the first player of the game.

Theorem 5.6. *For any position $S = [p, q]$ where $gcd(p, q) = 1$, $m(S)$ is the only move that is available and does not eliminate any other move.*

Proof. By the Chicken McNugget theorem, $m(S) = pq - p - q$ when $gcd(p, q) = 1$.
Suppose one available move $a < pq - p - q$ exists that does not eliminate any other available moves. Without loss of generality, we suggest that $p < q$.

We will first show any move b with $pq - 2p - q < b < pq - p - q$ is not available. According to Chicken McNugget theorem, exactly one move among $k, pq - p - q - k$ is available. Since any move $k < p < q$ is obviously available, any move b is not available. If $a < pq - 2p - q$, a will eliminate $a + p < pq - p - q$, an available move (since a is an available move). Hence $m(S)$ is the only available move that will not eliminate any other available move. □

Theorem 5.7 (Winning first move)**.** *The first player can always win by picking the first move p, where p is any prime number greater than* 3.

Proof. This statement is equivalent to the statement that position $[p, q]$ where p is a prime > 3 and q is any legal move after p is always a $\boldsymbol{N}$-position 1. This is obvious, since $gcd(p, q) = 1$ upon any available move q. □

Notice that for $\boldsymbol{N}$-position 1, the existence of largest available move at the position $S = [x_1, x_2, ...x_n]$, $m(S)$ is the pre-requisite condition. This requires that the number of available moves is finite, i.e $gcd(x_1, x_2, ...x_n) = 1$.

This provides us the idea of dividing the positions in the game the Sylver Coinage to two types.

- type 1: $m(S) = \infty$ In stage 1, $gcd(x_0, ..., x_n) > 1$, the number of available moves are infinite. In this case, it is not possible to use an exhaustive listing to find the winning move.
  For example, $S = [5]$, $S = [4, 6, 14]$ are positions
- type 2: $m(S) \neq \infty$ In type 2, $gcd(x_0, ..., x_n) = 1$, the number of available moves are finite (bounded by $m(S) - n$). In this case, it is possible to determine the winning strategy by an exhaustive listing of move choices.

For every single completed game, the positions must experience a transition from type 1 to type 2.It is obvious that the transition must only happen once.

Define move t as the transition move of position S if and only if t and S satisfies the following condition: $m(S) = \infty$; $m(St) \neq \infty$. We denote it as $t(S)$.

In the last part of the paper, we will prove a few not so obvious positions to be $\boldsymbol{N}$ or $\boldsymbol{P}$ using our previous results.

Theorem 5.8 (losing moves: 2,3)**.** *The player who first makes the move from* $[2,3]$ *loses.*

Proof. After the previous player makes the move 2 or 3, the next player will win by play 3 or 2 (since the moves 3 and 2 would never be eliminated by any possible other move, it is always available). By Chicken McNugget theorem, moves $[2,3]$ will eliminate every move except for 1. □

Theorem 5.9. *Position* $[2,3], [4,6], [6,9]$ *are* $\boldsymbol{P}$ *positions.*

Proof. For the position $[2,3]$, it is obvious that the only "available" move is 1, which is a losing move. Although this seems trivial, it will lead to the lemma:

Pre-existing moves $[2p, 3p]$ will eliminate every move divisible by p itself.

Hence, for position $[4,6]$, the next player must choose 2 (losing move) or an odd move. Let this move be t, $t = t(S)$ and the position is from now on stage 2.

The two moves $[4,6]$ will together act similarly to the move 2. The only difference is that it will not give the factor 2 itself.

Hence, by taking the move $2k+3$, the previous player can arrive at the position $[4,6,2k+1,2k+3]$ which has the same set of available move as position $[2,2k+1]$ except for the move 2, which is a losing move before the move 3 is taken. Hence, the previous player can follow exactly the same winning strategy of position $[p,q]$ $(gcd(p,q) = 0)$ If the next player does not take the move 3 or 2. Otherwise, the previous player can always win by immediately taking the corresponding move 2 or 3.

Similarly, $[6,9]$ is also $\boldsymbol{P}$ position. □

Theorem 5.10. *Position* $[x_1, x_2, ..., x_n]$ *are* $\boldsymbol{N}$ *positions if* $gcd(x_1, x_2, ..., x_n) = k$, $p|k$ *for prime number* $p > 3$.

Proof. The next player win immediately by taking the move p. The position $[p, x_1, ..., x_n]$ is equivalent to the position $[p]$, which is a $\boldsymbol{P}$ position. □

Theorem 5.11. *Position* $[4, 4k+2]$ *is a* $\boldsymbol{N}$ *position for* $k > 2$.

Proof. The next player can win by taking the move 6, so that the position is equivalent to $[4,6]$. □

Theorem 5.12. *Position* $[8, 12]$ *is* $\boldsymbol{P}$ *position.*

Proof. After $[8, 12]$ are taken, every move divisible by 4 is eliminated except for 4.

If the next player take the move 4, the position $[4, 8, 12]$ is equivalent to $[4]$. The previous player can win by taking 6 immediately to make the position equivalent to $[4, 6]$ which we have already proven to be a $\boldsymbol{P}$ position.

Similarly, if the next player take the move 6, the previous player can win by taking 4.

If the next player take move $2k + 1$, it is obvious (by Chicken McNugget theorem) that the new position, $[8, 12, 2k + 1]$ has 4 more available moves, $[4, 2k + 5, 4k + 6, 6k + 7]$ compared to the $\boldsymbol{N}$ position $[4, 2k + 1]$. The previous player can win by taking the move $2k + 5$, and then follow an "anti-strategy stealing argument": Suppose the move 4 is not available. In this situation, if the next player can win by taking the move $6k - 1$ (which is $m(S)$), the previous player can take 4 to steal his winning strategy, as no other moves except for 4 is eliminated after taking 4. Similarly, if the next player take 4, the previous player can take $6k - 1$.

If the next player take the move $4k + 2 \neq 6$, the new position $[8, 12, 4k + 2]$ has 2 more available moves, $[4, 4k + 6]$ compared to the $\boldsymbol{N}$ position $[4, 4k + 2]$. The previous player can win by taking the move $4k + 6$, and then follow the same winning strategy of "anti-strategy stealing argument". $\square$

6 Conclusions

In this paper, we researched on two classical number theory problems: the Frobenius Coin problem and the Sylver Coinage using algebra and number theory approaches.

In the first stage of our research, we have come up with a general algorithm to compute the Frobenius number, as well as algorithms with less time complexity for special set of numbers.

In the second stage of our research, we have re-visited the classical result of prime winning openings, as well as proved a few positions to be $\boldsymbol{N}$ or $\boldsymbol{P}$ positions.

In conclusion, despite this being a notoriously difficult problem for mathematicians to crack, we hope that our research has helped to gain some insight on the structure of both of these problems and will help develop some background and ideas for future work.

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