# How to get **NIM-ble**

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# (1) Introduction and rationale

## 1.1 What is the game we are researching?

The game we are going to research on is Nim, a game which is said to have originated in China—it closely resembles the Chinese game of 捡石子, or "picking stones".

## 1.2 So what is Nim?

**Nim** is a mathematical game of strategy in which two players take turns removing (or "nimming") objects from distinct heaps or piles. On each turn, a player must remove at least one object, and may remove any number of objects provided they all come from the same heap or pile. Depending on the version being played, the goal of the game is either to avoid taking the last object or to take the last object.

## 1.3 Why Nim?

Elliot Ng

For me, I have tried playing Nim with others since a very long time ago. However, 90% of the time, I seem to lose to them. And as for the other 10%, it was probably due to luck. Hopefully, by finding strategies for Nim, I will be able to win them more often, by skill rather than luck.

Bryan Low

I have played Nim before, but at that time it was purely a game that I played for fun with my friends. However, after I learnt that there were mathematical concepts to the game, and strategies to win or place yourself with the best chance of winning, I felt that this game was indeed very interesting and I wanted to find out more about the true mechanics of the game.

Ryan Soh

I have not played this game before. After learning how to play this game, I became very interested in the game. While this game may look simple, I realised that there is much more to the game. I hope that in this project, I can not only find out strategies about the game, but also find out why the strategies work.

Pei You

Like Elliot, I have tried playing Nim with others since I was young. I found it pretty simple and interesting. Hopefully, by finding strategies for Nim, I will be able to understand the small things about it.

# (2) Literature review

## 2.1 History of the game

Variants of Nim have been played since ancient times.[1] The game is said to have originated in China—it closely resembles the Chinese game of 捡石子 *jiǎn-shízi*, or "picking stones"[2]—but the origin is uncertain; the earliest European references to Nim are from the beginning of the 16th century. Its current name was coined by Charles L. Bouton of Harvard University, who also developed the complete theory of the game in 1901,[3] but the origins of the name were never fully explained.

## 2.2 Rules of the game

The game is between two players and is played with either 2, 3 or 4 heaps of any number of objects. The two players alternate taking any number of objects from any one of the heaps. The goal is to be the last to take an object. In misère play, the goal is instead to ensure that the opponent is forced to take the last remaining object.

### 2.2.1 Example of a game of Nim

A normal game played between fictional players Bob and Alice, who start with heaps of 3, 4 and 5 objects. (not misere play).

| Size of heaps | | | Move |
|---|---|---|---|
| A | B | C | |
| 3 | 4 | 5 | Bob takes 2 from A |
| 1 | 4 | 5 | Alice takes 3 from C |
| 1 | 4 | 2 | Bob takes 1 from B |
| 1 | 3 | 2 | Alice takes 1 from B |
| 1 | 2 | 2 | Bob takes entire A heap |
| 0 | 2 | 2 | Alice takes 1 from B |
| 0 | 1 | 2 | Bob takes 1 from C |
| 0 | 1 | 1 | Alice takes entire B heap |
| 0 | 0 | 1 | Bob takes entire C heap and wins |

### 2.2.2 What strategies are there currently?

According to Wikipedia, Nim has been mathematically solved for any number of initial heaps and objects, and there is an easily calculated way to determine which player will win and which winning moves are open to that player.

The key to the theory of the game is the binary digital sum of the heap sizes, i.e., the sum (in binary), neglecting all carries from one digit to another. This is known as nim-sum.

Example, if I have 3 objects in heap 1, 4 objects in heap 2, 5 objects in heap 3, then the nim sum will be:

| | | |
|---|---|---|
| $011_2$ | $3_{10}$ | Heap A |
| $100_2$ | $4_{10}$ | Heap B |
| $101_2$ | $5_{10}$ | Heap C |
| Total: $010_2$ | $2_{10}$ | |

Since 10 in binary is the equivalent of 2 in the present numeral system, the nim-sum is 2. The strategy states that the nim-sum should be 0 after a player's turn, for that player to win (that is for normal play, when the last player who takes an object wins.)

In the event that Nim is played as a misère game, in which the player to take the last object loses, the nim-sum should still be zero, until the number of heaps with at least two objects is exactly equal to one. Then, remove the objects in the heaps such that it is equal to one or zero (depending on the number of heaps that have objects; if it is odd, then remove until zero, if it is even, remove until one). By then, there is a safe path to winning.

### 2.2.3 The 21 Game

The goal of the game is to **NOT (**or to be, depending on the type of play) be the first person to say "21". The rules are that you can only add 1,2 or 3 to whatever the other player says.
For example, the first player can say "1", "2" or "3". If he said "2" the second player can say "3" ,"4" and "5"and so on.
This game is similar to Nim as the goal is to prevent yourself from reaching (or aim to reach) a specific number, '21' in this case and '0' for Nim.

# (3) Research Questions + Solutions and Explanations

Since there is already a proven strategy as to how to win the normal game, we did not focus on that, but rather on how that strategy changes when we alter the rules slightly.
Hence, the 3 research questions are as follows:

## 1. How will removing the same number of objects from all the piles on every turn affect the game?

### Definition of the rules:

First and foremost, for this research question, only **misere play** is supported, since this is an adaptation of a game that states that it has to be misere play all the time.
Every time I remove a certain number of objects from one pile, the same number of objects is removed from each pile. For instance:

| Pile | A | B | C |
|---|---|---|---|
| | 9 | 5 | 8 |

If I choose to remove 3 from each pile, it becomes:

| Pile | A | B | C |
|---|---|---|---|
| | 6 | 2 | 5 |

However,no more than 5 objects can be removed, since Pile B will have a negative number of objects, which is against the rules.

### Solution:

Let’s assume that the object with the smallest number of piles has n. Here, we discuss a few cases.

### Case 1: There is only one pile.

In this rather straightforward case, there are 2 cases to discuss.

**Case 1.1: You can subtract any number you desire.**

In this case, the player to start first wins!

**Case 1.2: There is a limit to how much you can subtract.**

Let the maximum number of objects you can subtract be k. In this case, if n gives a remainder of 1 when divided by k+1, whoever starts first loses. Otherwise, the player to start first wins. But why is that so?

First, let us define some terms:

- A **winning position** is a position whereby the person to move can win.
- A **losing position**, on the other hand, is a position whereby the person to move cannot win.

In this case, 1 is a losing position, and 2 to k+1 are all winning positions for obvious reasons. However, k+2 is a losing position since no matter how a person subtracts, he gives the other player a winning position. Then, k+3 all the way to 2k+2 are all winning positions since the player to move can subtract such that n=k+2, which is a losing position for the opponent player. And then again, 2k+3 is a losing position, for the same reasons as k+2. If the process is repeated several times, you would realise that every number that has a remainder of 1 when divided by k+1 is a losing position. Hence, the problem is solved.

## Case 2: There is more than one pile.

Solution:

For piles N, N+A, N+B, N+C.... among N, A, B, C....., if there is an even number of losing positions, we should start first. Otherwise, we start second. That is the winning strategy.

### But why is that so?

If we look at the previous definition, a **winning position is one whereby the player who starts first wins.** Following that logic, the number of winning positions doesn't matter, since the first player will always win. However, **a losing position is a position whereby the player who starts first loses, and hence the player who starts second wins.** Thus, following that logic:

- 1 losing position means that the second player wins.
- 2 losing positions means that the first player wins...

And if we keep following that logic, we come to the conclusion that if there is an even number of positions, the first player will win. Otherwise, the second player wins.

### Case Study:

Again, solving this case requires the use of winning and losing positions.

Let's say that we have 2 piles, and the situation is as follows:

| Pile | A | B |
|---|---|---|
| | N | N+H |

And that I can subtract any number from 1 to k.
We have 3 cases to discuss.
**Case 2.1: Both N and H are winning positions i.e.when divided by k+1, does not have a remainder of 1.**
In this case, it is better to start first. N being the winning position means that the second player is the last to move before pile A is cleared, which means that the player starting first will get to start subtracting from H first, which is a winning position for him.
**Case 2.2: Only one of N and H are winning positions.**
If N is a losing position, it will be the first player who touches pile A last, and the second player gets to subtract from H first, which is a winning position.
Similarly, if N is a winning position, it will be the second player who touches pile A last, and the first player gets to subtract from H first, which is a losing position for the first player. Therefore, it is better to start second.
**Case 2.3. Neither of N and H are winning positions.**
Since N is a losing position, it will be the first player who touches pile A last, and the second player gets to subtract from H first, which is a losing position for the second player. Hence, it will be better to start first.

From the case above, it can be seen that when there are 0 or 2 losing positions among N, H, we should start first. Otherwise, we should start second, thus further supporting the above claim.

## 2. How will adding objects to the pile affect the game?

### Solution:

As it turns out, adding objects does not affect the game. But first, there are 2 problems that need to be solved, namely:

i. If one guy adds a certain number of objects, if the other guy subtracts the same number, the game is back to square 1.
ii. If both people keep constantly adding objects, then the game will never end. Thus, from ii, there has to be a rule such that each guy can only add objects a finite number of times per game.

But ultimately, that does not affect the game. However, why is this so?
**Case 1: You are only allowed to add objects 1 time per game**

If you go back to an earlier explanation, you will find out that when the nim-sum is 0, there will be no way to subtract such that the nim-sum remains 0. Similarly, it will be impossible to add.
When the nim sum is not 0, however, it is possible to subtract such that it becomes zero, thus winning the game. Similarly, adding will give the same result.
**Case 2: You can add objects more than once per game.**
In this case, adding objects 2 times, **CONSECUTIVELY** can actually affect the game. Here's the analogy:

- It's my turn. The nim-sum is zero. I add once, the nim-sum is no longer zero. I add another time, I end my turn with the nim-sum being zero, which does not change the outcome.

However, the problem lies within the fact that a player can only add/subtract once per turn, as per the rules. Also, even if there is a rule that I can add/subtract twice or more per turn, the opposite player can do the same. Using the above example:

- The opposite player can do the same as what I did to ensure that he has a nim-sum of 0 right after his turn.

Hence, adding DOES not affect the game.

## 3. The 21 Game: An Adaptation of Nim: What is the general formula for any number?

### Solution:

There are 4 cases to discuss, and for all of them, let's assume that we are playing the M game, M being the number that you should avoid or aim for (depending on the type of play), and that we can add any number from 1 to k, with k being the maximum number you can add on each turn.

Firstly, let's discuss normal play, whereby the aim is to reach M.

**Case 1: M is divisible by 1+k.**
In this case, the solution is pretty straightforward. **A player should start second**, and keep adding such that after every turn, the total number is a multiple of 1+k. For example:

- If a guy starts first and adds 2, I should add k-1 objects.
- If a guy adds 5 objects, I should add k-4 objects.

**Case 2: M is NOT divisible by 1+k.**

This case is also quite straightforward, albeit not so much compared to Case 1. Unlike Case 1, this case requires us to start first. Then, find the remainder when M is divided by 1+k. That will be the number we will add on the first turn. For instance, if I play:

- The 27 Game, and I am allowed to add from 1 to 6, I divide 27 by 7 (or 1+6), and get a remainder of 6. Hence, on the first turn, I would add 6.

Afterwards, it will become the same as Case 1. Using the above example, after the first turn, it becomes:

- The 21 Game, and I am allowed to add from 1 to 6. I am moving second.

Thus, the problem is solved.

Now, we assume that we are playing misere play, in which the goal is to NOT reach M. For both cases below, the only way to force the other player to reach M is to get to M-1, hence the cases we are going to discuss will be regarding M-1 rather than M.

**Case 3: M-1 is divisible by k+1.**

This case is also quite straightforward. Like Case 1, we start second, and after every turn, the cumulative frequency should be such that it is a multiple of 1+k.

**Case 4: M-1 is NOT divisible by k+1.**

This case may seem to be the least straightforward of all, but in reality, it has a solution similar to that of Case 2. Like Case 2, we have to start first. Then, we take the remainder of M-1/k+1. That will be the number we add on the first turn. Afterwards, it becomes similar to Case 3.

## (4) Some extra things we found along the way

- (Related to Q1) What is the probability of starting first and being in a winning position, assuming that there is only one pile and the maximum we can subtract is k.

  **Solution:**

It is k/k+1. Now imagine being in a losing position. That will be sad.

- (Related to Q1) Is 0 a winning position?
  **Answer:**
  Yes, it is.

## (5)Intended Methodology

Due to the COVID-19 pandemic, meeting in person will be very hard, so we plan to meet online to discuss and complete our project.

## (6) Intended Timeline

Initial Timeline (planned):
**January: Formation of group**
**late-March: Completion of written report**
**Early-April: Completion of proposal evaluation slides.**
**April-July: Completion of initial draft of final evaluation slides.**
**Post-Mid-term evaluation: Final touches.**
Final Timeline( which was affected by COVID-19):
**January: Formation of group**
**late-March:Completion of proposal evaluation slides.**

**April-August: Final evaluation slides and written report complete (since mid-term evaluation was cancelled)**

## (7)References

Nim - Wikipedia
https://en.wikipedia.org/wiki/Nim