8-11

Traffic Optimization - A Study of Singapore’s Traffic Flow

Written Report

<u>Group Members:</u>
Jin Zhijia 3i113 (Leader)
Samuel Liew Rui 3i125

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1. Introduction

The Problem at Hand

Traffic congestion has proved disadvantageous to Singapore. Economically, traffic congestion wastes people's time that can be spent on work instead, slowing down people's productivity and Singapore's economic growth as a whole. Environmentally, it not only degrades air quality, causing air pollution, but also causes noise pollution. Furthermore, traffic congestion may even affect people's mental state, whether be it due to the disruptive noises or pollution that arise from traffic jams or the stress and worry that they will arrive at their destination late. All in all, traffic congestion is a bane in many aspects.

Rationale

It is evident that traffic congestion is a major problem in Singapore that, if improved, would bring about much benefit to the nation. Thus, we approach this topic with a hope of understanding and improving the current traffic system in Singapore, ultimately reducing traffic congestion for smoother journeys and minimizing the problems cause by traffic congestion.

Objectives

1. Explore the Mathematics behind traffic congestion.

2. Understand the underlying problems in Singapore's traffic system.

3. Formulate solutions for mitigating traffic congestion in both streets and highways.

4. Improve traffic conditions on roads, not just in Singapore, but in other cities as well.

Research Questions

1. What is an ideal traffic light set-up (on streets)?

2. How do we optimize a road network?

3. How would our ideal models function in real world situations?

Field of Mathematics

Our scope of study mainly focuses on the following fields of Mathematics:

1. Calculus
2. Graph Theory
3. Data Analytics

Terminology

Terms	Definitions
Traffic Density	No. of vehicles per unit length
Average speed	Average distance covered by vehicles per unit time
Traffic Flow	Number of vehicles passing a reference point per unit time
Dijkstra’s Algorithm	Algorithm for finding the shortest path between nodes on a graph
Dirac impulse	A function that is 0 everywhere except at certain points

2. Methodology and Timeline

Our Approach

To begin our investigation, we did thorough research in order to properly understand the mathematics behind traffic congestion thoroughly. We went online to read up on different research papers that provided us with key insights on not just the mathematics behind traffic flow, but also different perspectives on how to analyse and improve traffic systems (see Literature Review pg. 7). Apart from this, in order to visualise and implement our mathematical solutions into the real world, we developed a mathematical model encompassing traffic, and also went to different heavily congested roads in Singapore and observed the different traffic patterns on these roads as well as what could be the factors that caused traffic congestion.

Work Distribution

	RQ 1	RQ 2	RQ 3	Data	Report	Slides
Zhijia	⋆	⋆	⋆		⋆	⋆
Samuel	⋆		⋆	⋆	⋆	⋆

Timeline

Time Frame	Objectives
T1W4-8	Conduct preliminary research on the issue
March holidays	Read up on papers and existing research
T2W1-4	Proposal Evaluation
T2W5-7	Brainstorm suitable variables and approaches to our project
T2W7-10	Begin work on model
June holidays W1-2	Fine tune existing work, carry out further research to improve understanding on the matter
June holidays W3-4	Collect and collate data to evaluate the traffic situation in Singapore
T3W1-2	Perform thorough analysis on data
T3W2	Finalise model
T3W3	Answer RQ1
T3W4	Answer RQ2, begin written report
T3W5	Answer RQ3, begin final evaluation slides
T3W6	Finalisation and submission of written report
T3W7	Finalisation of final evaluation slides

3. Literature Review

In order to complete our project, we read up extensively on calculus and traffic modelling in order to formulate comprehensive solutions on how to reduce congestion. Below are some of the resources that we found extremely useful:

Resources

1. US Department of Transport - *"Traffic congestion and reliability: Trends and advanced strategies for Congestion Mitigation"* (2020)

Fig.1 shows the factors that caused traffic congestion during the 10 worst days of traffic in Washington D.C. From Fig.1, we see that traffic congestion spirals around three main factors, with High Demand (6 days) being the most consistent, then Bad Weather (5 days), and finally Incidents (3 days). It is to be noted, however, that Bad Weather and Incidents are factors that cannot be controlled. This source thus affirmed our decision to focus on improving the traffic system so as to be able to accommodate for the High Demand effectively.

	High Demand	Bad Weather	Incident
Day 1		*	
ThisDay 2	*	*	*
Day 3		*	
Day 4	*		
Day 5			*
Day 6	*	*	
Day 7	*		*

Day 8	*		
Day 9		*	
Day 10	*		

Fig. 1

2. University of California, Los Angeles - "A Mathematical introduction to Traffic Flow theory" (2015)

This article introduced the approach of analyzing traffic congestion from a microscopic and a macroscopic view, and taught us how to transit from one perspective to the other, for example using the Kernel Density Estimation (Microscopic → Macroscopic) It also taught us how to derive relationships between simple variables to come up with a more complex and appropriate model. Lastly, this paper provided us with ways to measure and study real world data to apply to our investigation and solutions.

3. SISTA, Department of Electrical Engineering - *"Optimal Traffic Light control for a single intersection"* (1998)

In the article, The author proposes a mathematical traffic light model, which relates average arrival and departure rates to optimal traffic light timings. This traffic light model involves a function λ_i comprising of a series of Dirac impulses which represent the arrivals of a car, to denote the average arrival rate at any lane L_i. Considering a cross-road set-up where i=1, 2, 3, 4, the traffic light model utilises basic calculus to calculate the optimal time t for t_{green} , t_{amber} and t_{red}. This is a good starting point for us to further develop a comprehensive traffic light model that is applicable in a real world context.

4. Institute of Physics, Humboldt University - *“Optimization of Road Networks Using Evolutionary Strategies”* (1997)

This article introduced another idea of optimizing overall traffic flow in Singapore, which is to optimize the road networks to reduce distances and time spent on roads. This allows for a more efficient and reliable road network which reduces the chances of congestions occurring. The article uses the method of describing destinations via nodes, and the roads connecting them using lines. With this, the most efficient road network can be found using a potential function which calculates the “worth” of each road network, which will play a major and important role in minimizing the effects of traffic congestion.

4. Results and Findings

Research Question 1

What is an ideal traffic light set-up?

Traffic Light Arrangement

To answer the question *"What is an ideal traffic light set-up (on streets)?"*, we first need to consider the most optimal way in which the traffic lights can be situated. Obviously, traffic lights are only necessary when different roads intersect. This intersection can be classified into two different categories: the T-shaped intersection and the X-shaped intersection. We will first consider the T-shaped intersection.

In order to optimise traffic flow in a "T-shape" intersection as much as possible, we must first consider the directions of the two roads. *Fig. 2a* shows an example arrangement. First, notice that, regardless of direction, two traffic lights need to be placed, since some lanes will not be able to move when another lane is moving. Next, L-shaped "sub-intersections" (see *Fig. 2b*) are optimal, since a car can travel from one road to another without a traffic light. We have thus transformed this optimization problem (for traffic light arrangement) into one of fitting the most L-shaped "sub-intersections" into the intersection.

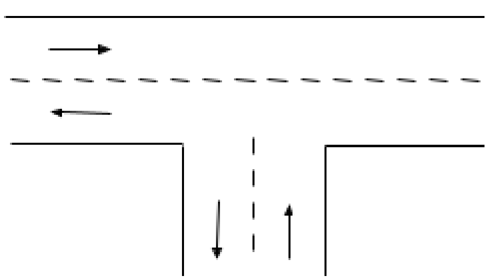


Fig.2a

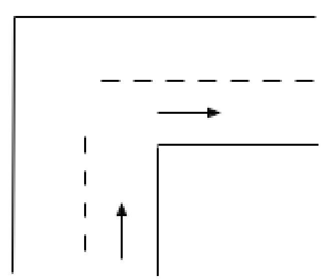


Fig. 2b

We now analyse *Fig. 2a*. Looking at all possible pathways that cars from one lane to another can take, we see that there are, in fact, no possible instances of an L-shaped "sub-intersection" in the intersection. *Fig. 2a* is thus not fully optimised. In contrast, however, by simply changing the direction of the roads, *Fig. 2c* provides a much more optimised set-up, with 2 possible instances of an L-shaped "sub-intersection" in the intersection.

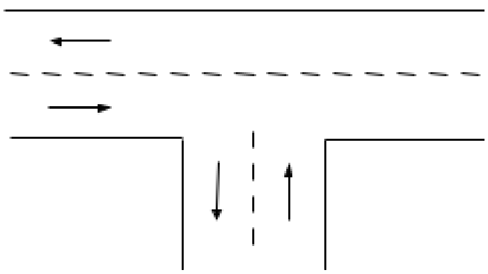


Fig. 2c

We now consider an X-shaped intersection. Using the same method of optimization as before, we exhaust all possible arrangements to find the most optimized ones (i.e. highest possible number of L-shaped “sub-intersections”). *Fig. 2d* shows one such arrangement, which has 4 possible L-shaped “sub-intersections”.

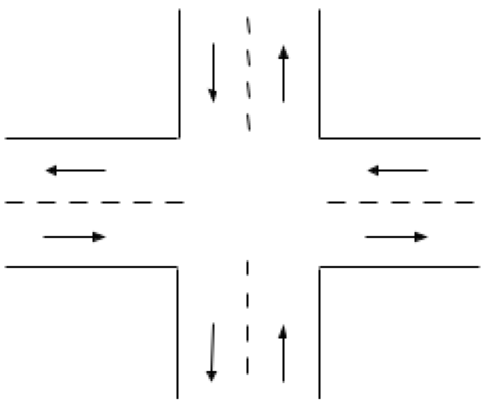


Fig. 2d

Possible limitations

One obvious flaw in this method of optimization, however, is that roads do not just function as pairs, but function as an entire system. Thus, for extremely complicated road networks like Singapore’s, it might not be very convenient to temper with the directions of the roads, since this would functionally change the entire road network of Singapore

Analysing a traffic light junction

Considering an intersection of 2 streets, let the roads leading to the junction be L_1, L_2, L_3 and L_4, and let the traffic lights at each road be T_1, T_2, T_3 and T_4. (Refer to Fig. 3a)

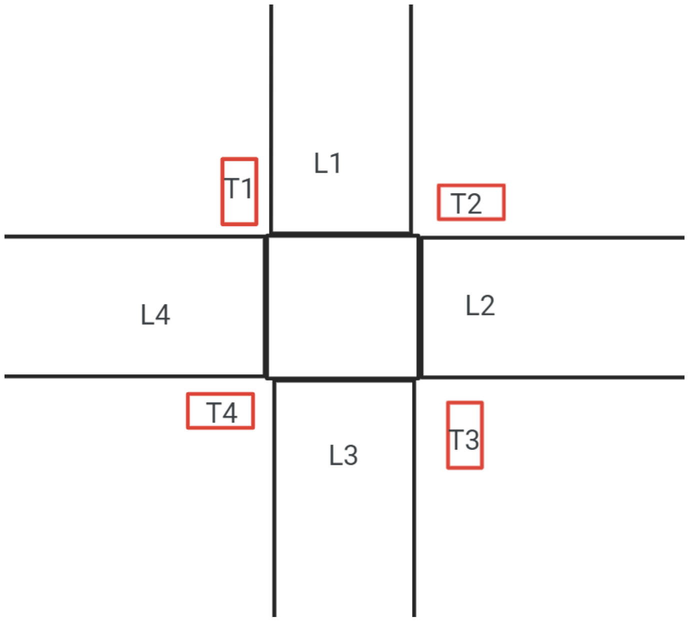


Fig. 3a

Each traffic light has 3 phases, red, amber and green. When the traffic light is green or amber, there are arrivals and departures from that lane. However, when the traffic light is red, there are arrivals but no departures. Let the number of vehicles on a road at a certain point in time be $L_i(t)$, the departure rates for lane L_i when T_i is green and amber be $\mu_i(t)$ and $\varphi_i(t)$ respectively, and let the arrival rates at any point in time be $\tau_i(t)$, which contains a series of Dirac impulses, where an impulse appears when a car enters, and where each impulse has an area of 1. Let the instants in time when T_1 and T_3 switch from amber to red be t_{2k}, and let that of T_2 and T_4 be t_{2k+1}, where k is an integer. Let the duration of amber be a fixed constant, σ_{amb}.

By defining these conditions, we can first better visualize the timings using a table (Fig. 2b)

Period	$\boldsymbol{T_1}$	$\boldsymbol{T_2}$	$\boldsymbol{T_3}$	$\boldsymbol{T_4}$
$t_0 \rightarrow t_1 - \sigma_{amb}$	red	green	red	green
$t_1 - \sigma_{amb} \rightarrow t_1$	red	amber	red	amber
$t_1 \rightarrow t_2 - \sigma_{amb}$	green	red	green	red
$t_2 - \sigma_{amb} \rightarrow t_2$	amber	red	amber	red
$t_2 \rightarrow t_3 - \sigma_{amb}$	red	green	red	green
...	**...**			

Fig. 3b

Hence, the rate of change of cars in a particular lane can be derived over the different time periods of one cycle (i.e. $t_{2k} \rightarrow t_{2k+2}$, where k is an integer). Let us take L_1 for example.

$$\frac{dL_1(t)}{dt} = \tau_1(t), \text{ when } t_{2k} \rightarrow t_{2k+1},$$

$$\tau_1(t) - \mu_1(t), \text{ when } t_{2k+1} \rightarrow t_{2k+2} - \sigma_{amb}, \text{ and}$$

$$\tau_1(t) - \varphi_1(t), \text{ when } t_{2k+2} - \sigma_{amb} \rightarrow t_{2k+2}$$

, where k is an integer.

This would imply that at the instants where the lights switch between red and green for L_1 , we have:

$$L_1(t_{2k+1}) = L_1(t_{2k}) + \int_{t_{2k}}^{t_{2k+1}} \tau_1(t)\, dt$$

$$L_1(t_{2k+2}) = L_1(t_{2k+1}) + \int_{t_{2k+1}}^{t_{2k+2-\sigma amb}} (\tau_1(t) - \mu_1(t))\, dt + \int_{t_{2k+2-\sigma amb}}^{t_{2k+2}} (\tau_1(t) - \varphi_1(t))\, dt,$$

where k is an integer.

Given these relationships, we are also able to estimate the queue lengths on each road.

Analysis based on frequencies of visits

Of course, different traffic junctions may be utilised by a different number of cars, depending on a location. According to the Land Transport Authority (LTA), many of the roads in Singapore have sensors beneath its roads to measure the average arrival and departure rates of a certain road. With this easily accessible data, it would be easy for the government to optimize the traffic lights of certain roads.

Firstly we must establish our approach in optimising traffic light timings. In our calculations, based on our observations, we will assume

$\mu_i (t) = -\dfrac{5}{\frac{5}{9 - \tau i\,(t)} + e^{-t+2.1}} - e^{-t+2.1} + 9$, σ_{amb} = 3s, and $\varphi i\ (t)$ = 0.5 (final $\mu_i (t)$, *ie* $\tau_i (t)$),

and we will assume the rates of arrival and departure are rates based on our observations instead of Dirac impulses. The graph of $\mu_i (t)$ is shown in Fig. 3c. As can be seen, at t=0, the number of cars leaving the intersection increases rapidly when the light turns green, reaching a peak, before settling down to the rate $\tau_i (t)$, assuming the arrival rate remains constant throughout.

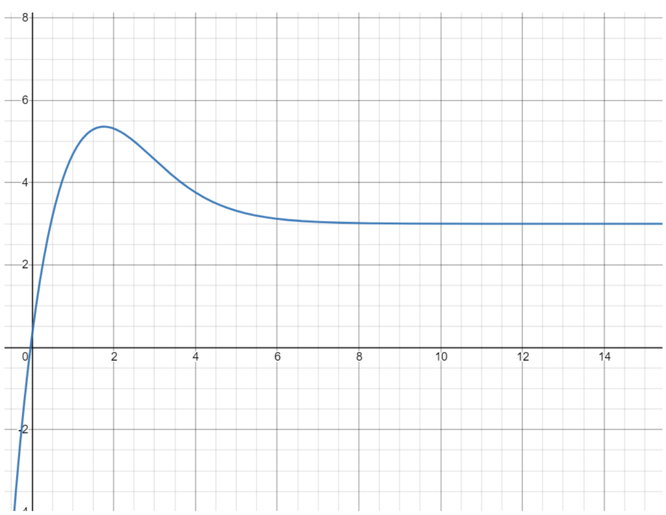


Fig. 3c

To optimize, we should try to maximise the total number of cars allowed to pass through the intersection in all lanes within one cycle. This can be calculated using our model that we have presented above, which will allow us to find the total number of cars in a lane at any point in time, and the total number of cars passing through the junction within one cycle. We can then maximise this number to find the optimal traffic light timing. Our findings will be shown later in Research Question 3.

Research Question 2

How do we optimize a road network?

Dijkstra's Algorithm

In order to optimise the road network, we first introduce Dijkstra's Algorithm, which is an algorithm for finding the shortest path between nodes on a graph. We will now use an example to demonstrate how Dijkstra's Algorithm works.

Suppose you are trying to travel from node T to Y.

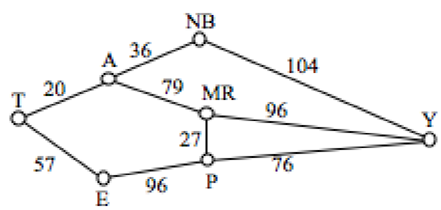


Fig. 4a

Step 1: Mark the ending vertex with a distance of 0 (see *Fig. 4b*)

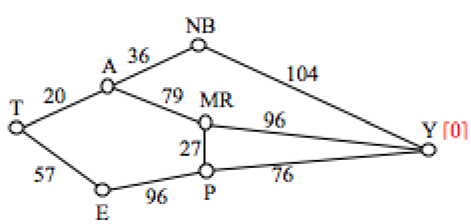


Fig. 4b

Step 2: Calculate the distance of the vertices leading to Y (see *Fig. 4.c*)

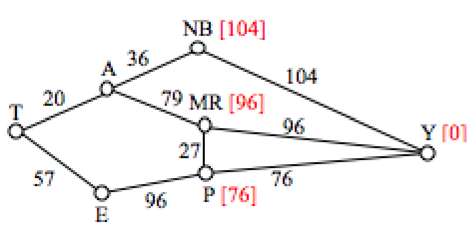


Fig. 4c

Step 3: For the vertices leading to Y (vertices NB, MR and P), repeat Step 2 (see *Fig. 4d*). Notice that vertices NB and MR share a vertex, A. In this case, the smaller computed value will be written.

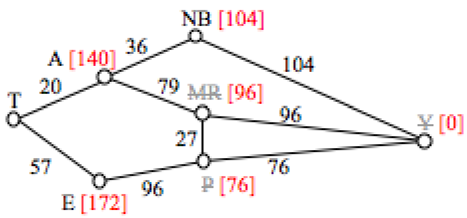


Fig. 4d

Repeating Step 3 again, we see that the path comprising vertices T → A → NB → Y is the shortest path. (see *Fig. 4e*)

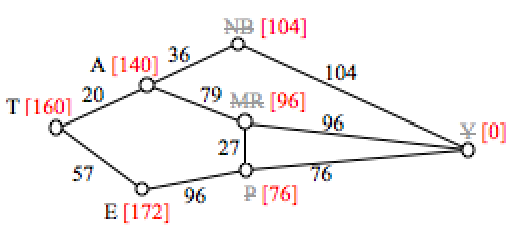


Fig. 4e

The Evaluation Function

When optimising a road network, they are mainly two criterias to consider:

1. To minimise mean detour taken for a road network
2. To minimise the cost needed to construct a road network

When solely length of detour is concerned, the network would look something like *Fig. 5a,* known as a direct link system.
When solely the cost of the network is concerned, the network would look something like *Fig. 5b,* known as a minimal link system.

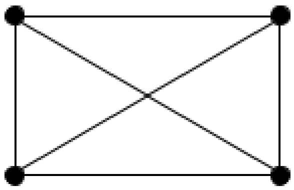


Fig. 5a

Fig. 5b

For practical applications, these two networks could be sufficient under certain circumstances. A direct link system (see *Fig. 5a*) could be appropriate when cost of construction is not of concern. Especially because of the high number of intersections between roads, overhead highways might be necessary, and this would certainly add to the cost of construction. A minimal link system (see *Fig. 5b*), on the other hand, could be appropriate if congestion is minimised and the connection distance can be passed with a very high speed.

However, in real life, governments will seek to balance out the two demands based on their circumstances. For example, a first-class country might seek to construct a well connected road network with little detour, so their concern on cost would be lower. A third-world country, in contrast, might be more conscious of its economical limitations, and would seek to construct a cheap and efficient, albeit less connected road network.

To start, let us represent a road network with a set of nodes p_1, p_2... p_n. We now propose a function to evaluate each road network:

$$U(g)=D(g)+C(g)$$

where *D(g)* is the total detour taken for the road network *g* while *C(g)* is the total cost needed to construct the road network *g*. Based on our function, an *optimal road network* would be one that minimizes the value of *U(g)*, all while being able to connect each node to every other node.

We first consider how *D(g)* can be computed. By definition, the detour taken by a car travelling from one node to another is the distance travelled by the car minus the shortest distance between the two nodes (i.e a straight line connecting the two nodes). So, the total detour is as follows:

$$D(g)=\frac{1}{2}\sum_{i,j=1}^{n} h_{i,j} - l_{i,j}$$

where $h_{i,j}$ represents the shortest *route* between two nodes, p_i and p_j, and $l_{i,j}$ represents the shortest *distance* between two nodes, p_i and p_j.

Note that with coordinates, the lengths of $l_{i,j}$ is just $\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ by pythagoras theorem, while the lengths of $h_{i,j}$ is just the smallest summation of multiple lengths between nodes in between the two nodes, which can be determined by Dijkstra's algorithm (see pg. 16-17).

Next, we consider how we would calculate the cost for constructing a road network. Obviously, the cost of constructing a road network is proportional to its length. Thus, we can normalize the cost such that it is equal to the length of the road network. The total cost is as follows:

$$C(g)=\frac{1}{2}\sum_{i,j=1}^{n} \lambda_{ij} l_{i,j}$$

where λ_{ij} is a conditional variable: λ_{ij}=1 when there is a *direct* connection between nodes p_i and p_j; λ_{ij}=0 when there is *no direct* connection between nodes p_i and p_j.

With that, our previous function can be rewritten into:

$$U(g)=\frac{1}{2}\sum_{i,j=1}^{n} h_{i,j} - l_{i,j} + \frac{1}{2}\sum_{i,j=1}^{n} \lambda_{ij} l_{i,j}$$

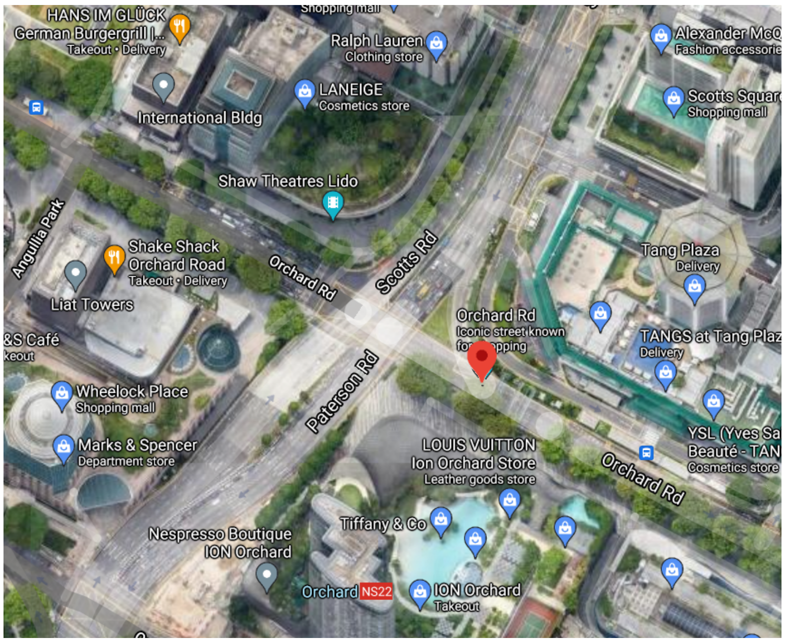
Given the coordinates of the nodes, this value can be easily computed, especially with a computer program. We have thus simplified this optimization problem into one which involves minimising the sum of detour and cost. We will now be showing a method of manually optimizing a road network with Singapore's road network in Research Question 3.

Research Question 3

How would our ideal models function in real world situations?

Our Traffic Light Model

In our example, the intersection we have selected is the Scotts Road - Orchard Road junction, as shown below:



This junction is visited by many cars, hence Orchard Road has a high average arrival rate of around 3.5 cars s^{-1}, while Scotts Road has a slightly higher arrival rate of 4.1 cars s^{-1}. We will assume the roads parallel to each other have a similar arrival rate.

Given that the previous traffic junction is 258.4m away from this junction, we should try to restrict the queue to about half of the road to ensure sufficient leeway for other cars approaching. Hence the queue that accumulates during the red light should be restricted to around 129.2m. If the average vehicle is around 4m long, the number of cars that should enter should be restricted to $32.3 * 4 \approx 129$, as there are 4 lanes. Hence, the maximum time should be restricted to about $129 \div 3.5 = 36.9$s (3sf)

Let the length of the green phase of the traffic light on Orchard Road be x s. We will also assume that the length of one entire cycle is 75.0 s.

Then, the total number of cars N allowed to cross the intersection in one cycle is given by the following equation

$$N = \int_{0}^{x} \left(- \frac{5}{\frac{5}{9-3.5} + e^{-t+2.1}} - e^{-t+2.1} + 9\right) dt + \frac{3.5}{2} \cdot 3$$
$$+ \int_{0}^{69-x} \left(- \frac{5}{\frac{5}{9-4.1} + e^{-t+2.1}} - e^{-t+2.1} + 9\right) dt + \frac{4.1}{2} \cdot 3$$

which is equal to

$$- \frac{11ln(e^{-x}(10e^{x} + 11e^{\frac{21}{10}})) - 7x - 11ln(11e^{\frac{21}{10}} + 10)}{2} - e^{\frac{21}{10} - x}(e^{x} - 1) + 5.25$$
$$- \frac{49ln(e^{-67}(49e^{\frac{10x+1}{10}} + 50e^{67})) + 41x - 49ln(49e^{\frac{21}{10}} + 50) - 2829}{10} + e^{x-66.9} - e^{\frac{21}{10}} +$$
$$6.15$$

By differentiating the equation and finding the value when the derivative is 0, we get x = 34.125, which gives rise to a red time of 72-34.125= 37.875. As this is only slightly higher than the maximum red time, no restrictions have to be made. Hence, by utilising sensors and an optimization system, the government can better optimize traffic light timings for all systems and roads across Singapore.

Our Road Network Optimization Process

Fig. 6a shows Singapore's road network. For simplicity's sake, we have randomly identified 6 nodes across Singapore, nodes A, B, C, D, E and F (see *Fig. 6a*).

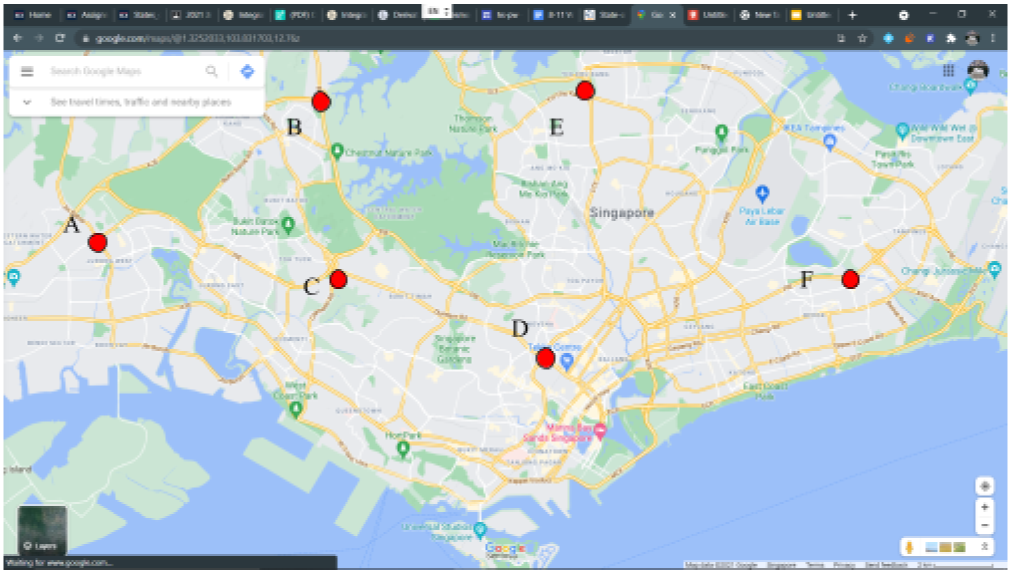


Fig. 6a

Step 1:
To start, we first connect each node to all other nodes and find the distance between any two nodes (see *Fig. 6b*).

With the help of Google Maps, the distance (in km) between every two node is as follows:

AB→ 9.17
AC→ 8.38
AD→ 16.02
AE→ 17.57
AF→ 27.32
BC→ 5.95
BD→ 11.81
BE→ 9.24
BF→ 20.79
CD→ 7.82
CE→ 10.47
CF→ 19.04
DE→ 9.15
DF→ 12.01
EF→ 12.49

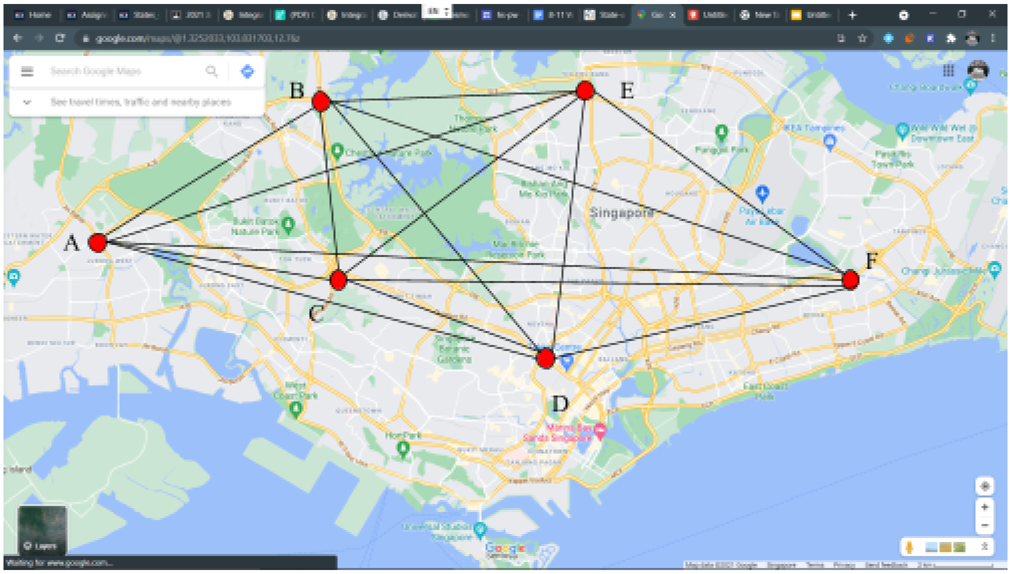


Fig. 6b

Using our previously found function, *U(g)* is simply the sum of all the distances between every 2 nodes. So *U(g)*=197.23. This is, in actuality, a direct link system.

Step 2:

We now begin to remove roads between nodes to try and minimize *U(g)*. Take note that by doing so, the detour will increase, while the cost will decrease. This removal should be systematic. Suppose we remove a road with length k. Then the cost will lower by k, but in return, the detour will increase by $h_{i,j} - k$, where as previously mentioned $h_{i,j}$ is the new shortest distance between p_i and p_j. Thus, we should only remove a certain road iff $h_{i,j} - k \leq k \Leftrightarrow h_{i,j} \leq 2k$.

For example, by removing the road connecting AF (see *Fig. 6c*), *D(g)* increases to 8.38+19.04- 27.32=0.10 while *C(g)* decreases by 27.32. Since 0.20≤ 27.32, removing AF would lower *U(g)* to 170.11.

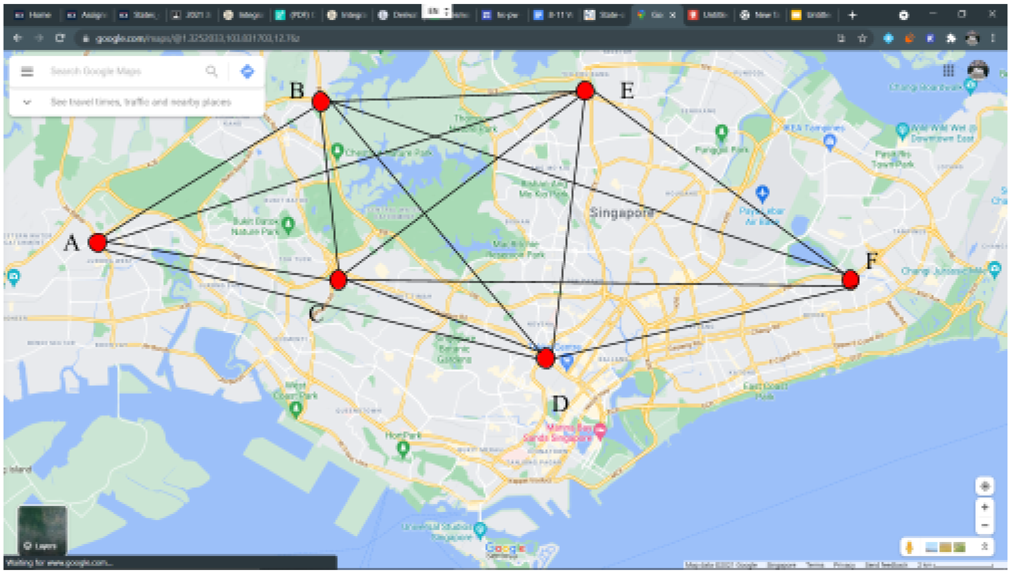


Fig. 6c

Step 3:
Repeat Step 2 until no further systematic removals can be performed. Using our process we were able to get a minimal *U(g)* value of 76.01 (see *Fig. 6d*)

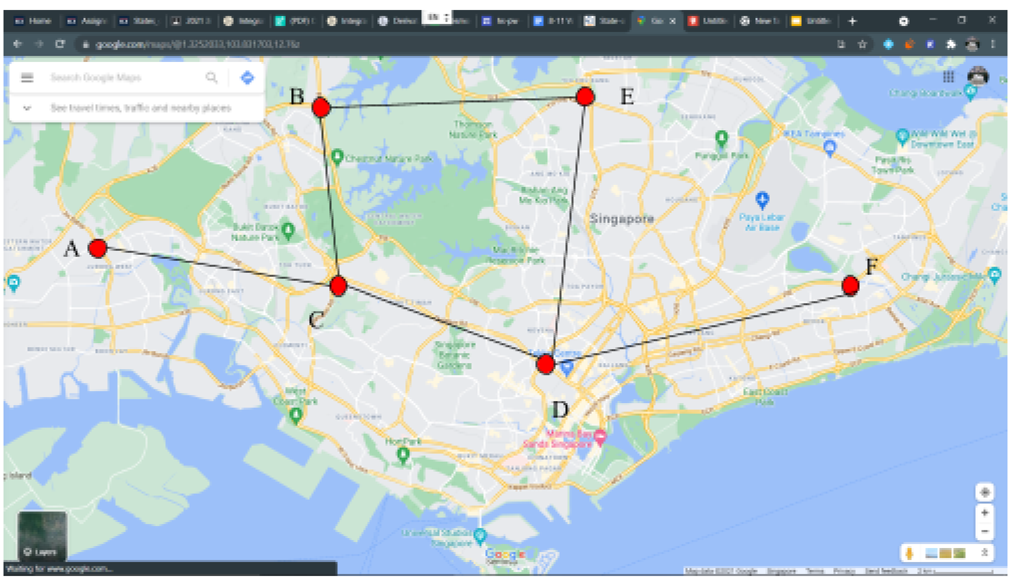


Fig. 6d

Examining the optimized road network with that of Singapore's actual road network, we see that there are some similarities, implying that Singapore's road network is actually fairly optimised. Notice however, that road BE might not be very realistic, since BE cuts right through the nature reserve of Singapore. This is thus one of our optimization process's limitations, that it is unable to properly take into account land constraints.

Hence, by utilising a similar approach to optimize a series of road networks, the government is able to increase connectivity between different nodes and allow from different routes to be taken, which would mean the congestion is spread out amongst these few routes, and an overall more reliable and efficient road network would mean less traffic congestion and improved traffic flow and speed.

5. Conclusion

Reflection and Learning Points

This project has allowed us to understand the severity of traffic congestion in Singapore, and how we can use mathematics to optimize traffic both on roads and highways. It has also allowed us to develop a greater understanding and interest towards math. However, it has also taught us important lessons:

1. Time management. This aspect of a project might seem small and insignificant, but it can really define a project's success or failure. Though our project was relatively smooth, there were certainly times when the lack of time management cost us our productivity in making progress. Especially in a time of crisis, we learned that with a good schedule and plan, half the battle is already won.
2. Communication. Good communication is key to a team's success. This was made especially obvious during this Pandemic, where we could rarely meet up physically to work on our project. Because of this we had to communicate efficiently and meet up online regularly to successfully get our work done.
3. Procrastination does us no good. Procrastination only pushes the workload behind.
4. A seemingly small problem could evolve into something critical in the future. We learned that we should always solve the problem at hand, and never leave something untouched.

Further Extensions

1. Regarding Research Question 1, some assumptions, such as consistent arrival and departure rates, were made in our calculations. We plan to use arbitrary values and rates as further research, perhaps using a computer simulation.
2. Regarding Research Question 2, we plan to look up deeper into the methods of road network optimization, to present a more general optimization process with less limitations.
3. Regarding Research Question 3, although our optimization process did provide us with an intended result, the manual calculation process is long and tiresome. Thus, we plan to create a computer program that would be able to calculate the values for us.

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