‘Squaring the Plane’

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1. Introduction

a. Objectives

In this project, we aim to:

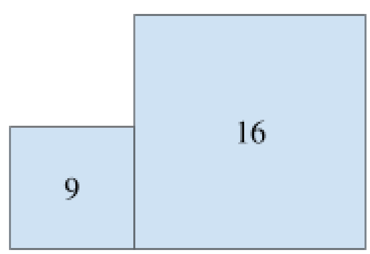
1) **Find out a (few) way(s) to tile a rectangular plane perfectly using distinct squares.**
2) **Find the possible patterns derived from our solutions.**
3) **Find out solutions for different variations of such a problem** by setting up different restrictions for this problem**.**

b. Description of Ideas

Our Math project is a math project about Tiling a Rectangular plane with Squares. In this project, we explore the possibilities of math and geometry, and aim to create and find planes that can be tiled by distinct squares.

One field of mathematics that needs to be used is geometry. As we are focusing on tiling a rectangular plane with distinct squares in this project, geometry must be used in order to prove certain statements within our project.

Finally, we shall determine the certain terms that would be used throughout the written report. One common phrase that would be seen is the phrase “perfect” plane. A “perfect” plane refers to such a plane that can be tiled by distinct squares of integer side lengths. Another common phrase that would be used throughout this paper is the word “Ell”. The word “Ell” means a six sided figure that is formed by 2 distinct squares. It can be seen in the diagram below, which is an Ell. (Fig 1.1)



(Fig 1.1)

c. Research Questions

1) Can **rectangular planes** be tiled by **distinct squares** of **integer** side lengths?

2) **If** there are such planes, then is there a **pattern to creating such planes**?

3) After looking through certain tilings of rectangular planes, is it also possible to tile a **square** perfectly with **smaller distinct squares** of **integer side lengths**?

d. Fields of Mathematics

1) Geometry
2) Algebra

e. Terminology

"Perfect Plane"	A plane that can be tiled perfectly by distinct squares with integer side lengths with no overlaps.
"Ell"	A six-sided figure formed by 2 distinct squares.

2. Literature Review

Our sources are:

Papenfus, M., 2021. *TILING RECTANGLES WITH SQUARES*. [online] Ime.math.arizona.edu. Available at: <http://ime.math.arizona.edu/ati/Math%20Projects/C1_MathFinal_Papenfus.pdf> [Accessed 24 July 2021].

Written by Melissa Papenfus, a teacher at the University of Arizona.

This research paper serves as a paper that brings us more in-depth knowledge of the topic, as well as also clearly explaining the certain methods that could be used to form a “perfect” plane, such as the algebraic method that shall be mentioned in the research questions.

BROOKS, R., SMITH, C., STONE, A. and TUTTE, W., 1940. [online] Carlo-hamalainen.net. Available at: <https://carlo-hamalainen.net/stuff/Brooks,%20Smith,%20Stone,%20Tutte%20-%20The%20dissection%20of%20rectangles%20into%20squares%20(1940).pdf> [Accessed 10 August 2021].

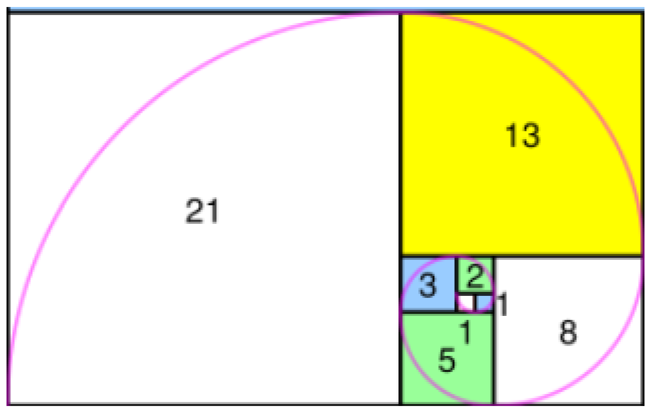
Written by R. L. Brooks, C. A. B. Smith, A. H. Stone AND W. T. Tutte, 4 mathematicians which closely studied this topic.

This paper mentions much that overlaps with the concepts of our project, from a simple method of tiling to Smith diagrams and high levels of Physics used to rigorously prove certain statements in the paper. Since it is very advanced, this paper was also mostly used for a slightly deeper understanding in this project and topic, with certain parts included within our project.

3. Study and Methodology

a. Methodology

Use the algebraic method to craft the specific planes. Implement the Fibonacci sequence and spiral to approach and tackle this question. The fibonacci method is a way of tiling that involves using the fibonacci spiral to arrange the squares in a certain method. A diagram showing this is seen below (Fig 1.2)



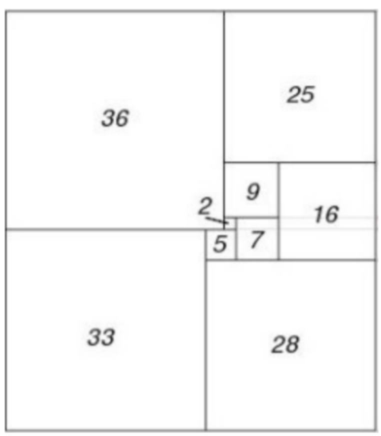
(Fig 1.2)

b. Research Question 1

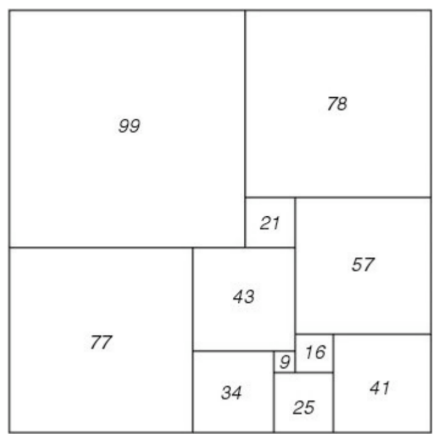
*<u>Can **rectangular** planes be tiled by **distinct squares** of **integer** side lengths?</u>*

Yes. Certain rectangular planes are able to be tiled by distinct squares. To form a rectangular plane that can be tiled, one is able to form an ell which will end up with a pattern filling up the whole rectangle.

Some examples of a rectangular plane being perfectly tiled by distinct squares are shown below.



(Fig 2.1)



(Fig 2.2)

c. Research Question 2

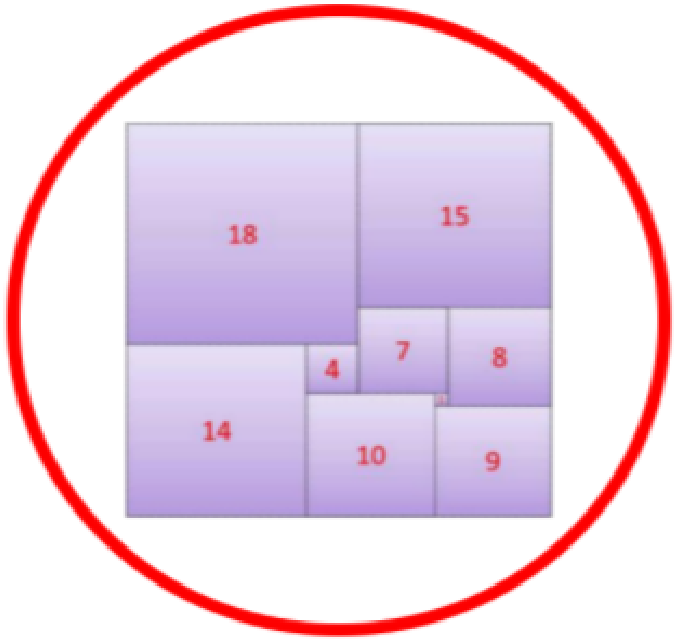
*<u>If there are such planes, is there a **pattern** to what planes are able to be tiled so?</u>*

Rationale:

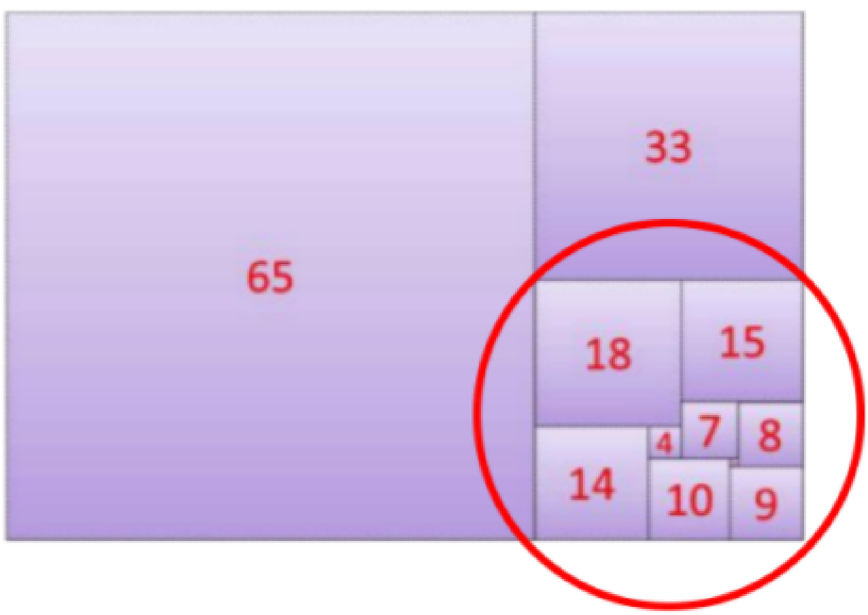
1. Tileable planes can be created when the Ell is maintained throughout the addition of squares until the last square is added to form the plane. To create a "perfect" plane that is tileable, firstly, we need to put together both the squares in an L-shape. Then, we shall put a square on the "flat side" of both squares, which is the side where both squares have their sides aligning to form a straight line. Although there is some trial and error, it is possible to form a plane as such.

2. Another possible way is by using a fusion of the Fibonacci method and the Ell method. First, using the Fibonacci method, tile the square fully and use the Ell to fill up the only square in the Fibonacci pattern which has appeared more than once. For example, by using the diagram of a square that has been squared by distinct squares, we can use such a square as the "base" of the Fibonacci tiling. (By replacing the repeated square with the tiling of this square.)

3. By figuring out one possible combination of squares that tile a plane, it is also possible to continue to infinitely create and tile other planes.



(Fig. 3)

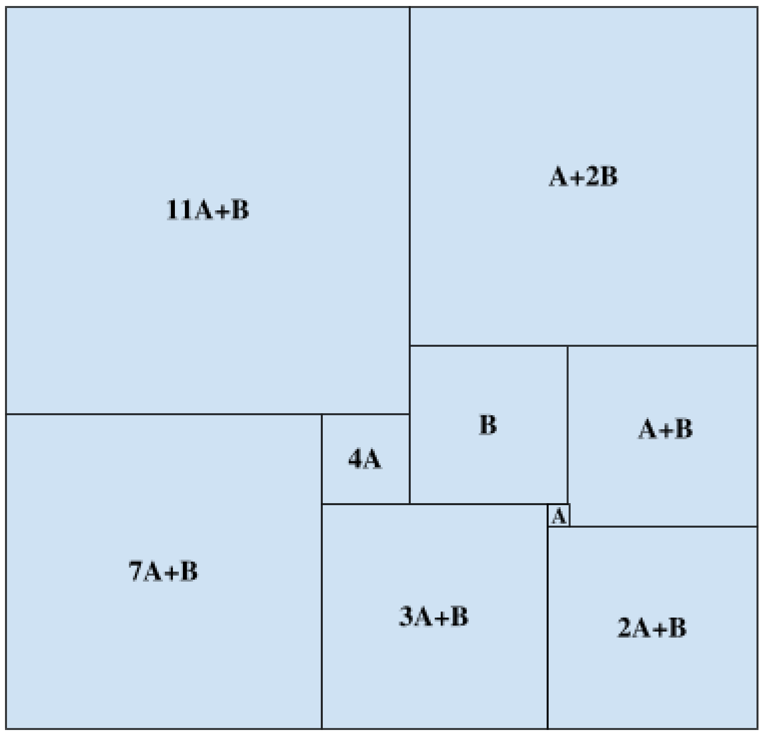


(Fig. 4)

We can compare the tiled planes as shown above. As shown, (Fig. 3) is formed by adding a square of one equal side length to the smallest possible tileable plane (Fig. 2), then connecting another square. It is also possible to infinitely add squares that share the same combined side length of the squares, to form new planes, similar to the Fibonacci method.

Another way to make a "perfect" plane is to roughly sketch a tiling system using Ell diagrams. Below is an example of how we attempt to do it.

In Fig 5.1, the lengths of the distinct squares which form the first Ell are A and B where A and B are arbitrary integers and B is larger than A.



(Fig 5.1)

Using the Ell formed by A and B, we develop more distinct squares until a rectangular plane is formed.

The width of the rectangle is 12A+3B and the length of the rectangle is 18A+2B. Because A and B are arbitrary integers, therefore the widths and lengths of the rectangle are also arbitrary and can change to fit the rectangle perfectly. In the figure (Fig 5.1), we can see that all the squares are distinct. Therefore, we can prove that a rectangular plane can be filled completely by multiple distinct squares. For easier understanding, we gave an example as shown in (Fig 5.2).

Here is a worked example:
Fill in the other squares in accordance to the lengths ‘A’ and ‘B’:

Now, compare the lengths of the left and right vertical sides of the large rectangle drawn. As shown in the diagram, the length of the left vertical side adds up to be (11A+B)+(7A+B) = 18A+2B, while the right vertical side adds up to be (A+2B)+(A+B)+(2A+B) = 4A+4B.

∵ left vertical side = the right vertical side

∴ 18A+2B = 4A+4B

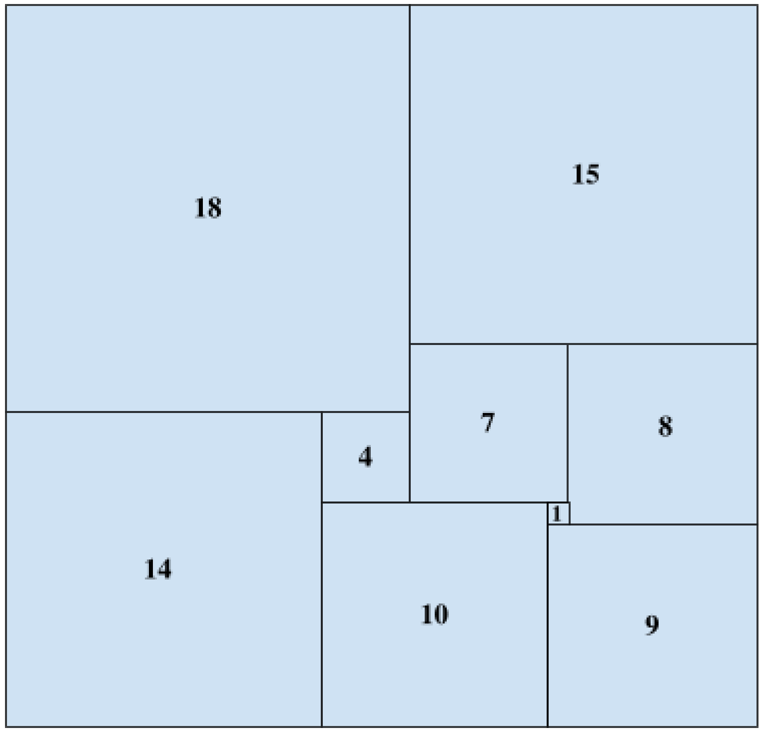
→14A = 2B

→ A = B

→ B : A = 7 : 1

We can observe that one pair of the opposite lengths is 32 and the other is 33. Therefore, these distinct squares form a rectangular plane. It adheres to our proof in (Fig 5.1).

Hence, the length of B is seven times that of A, thus a value could be replaced for these lengths, for example 1cm being the length of A and 7cm being the length of B.



(Fig 5.2)

d. Research Question 3

<u>After looking through certain tilings of rectangular planes, is it also -possible to tile a square perfectly with smaller distinct squares of integer side lengths?</u>

Rationale:

Through our extensive research, we have found that it is indeed possible to actually tile a square with smaller distinct squares, with each square being of integer side length. Although there is actually no fixed way at the moment to always form these types of squares, there are still some examples of such squares which include Figure 6.1, 6.2, and 6.3 of side lengths 112, 110 and 110 respectively.

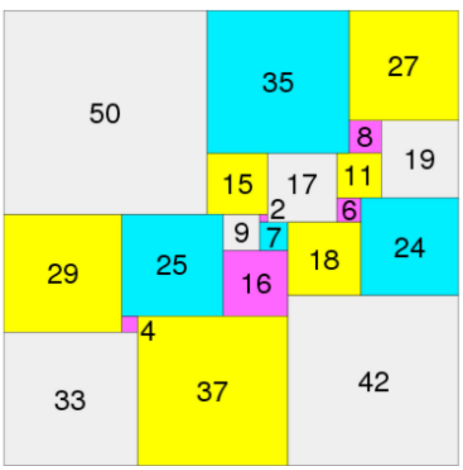


Fig 6.1 (112x112)

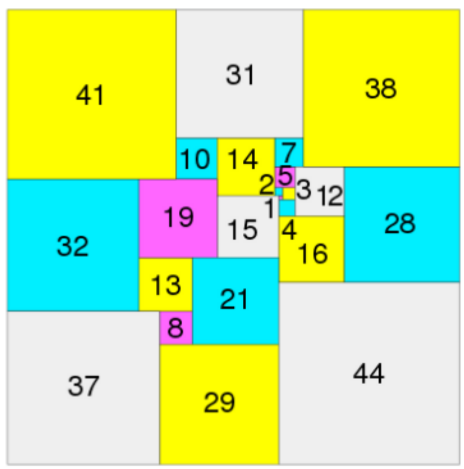


Fig 6.2 (110x110)

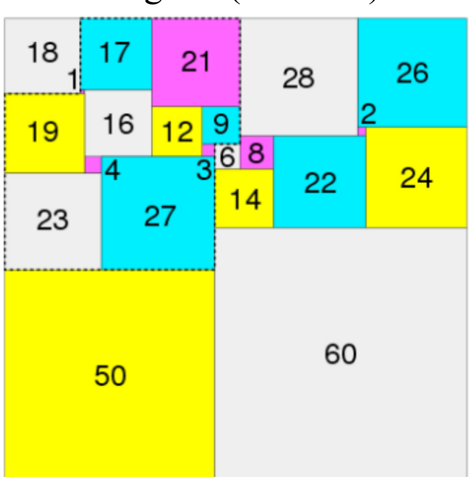


Fig 6.3 (110x110)

e. Conclusion:

For this project work, we have not managed to find many solutions to this 100 year old problem which has been puzzling scholars for generations. However, we still gained a deeper understanding of this topic. We have managed to turn out some very interesting solutions to this type of problem, such as by using the Ell followed by distinct squares to tile the whole rectangular plane. By using this solution, we can prove that a rectangular can be tiled perfectly by distinct squares. We also found out that the size of the rectangular plane controls the size of A and B used in the Ell mechanism.

References

1. Brooks, R., Smith, C., Stone, A. and Tutte, W., 2021. *THE DISSECTION OF RECTANGLES INTO SQUARES*. [online] Carlo-hamalainen.net. Available at: <https://carlo-hamalainen.net/stuff/Brooks,%20Smith,%20Stone,%20Tutte%20-%20The%20dissection%20of%20rectangles%20into%20squares%20(1940).pdf> [Accessed 10 August 2021].

2. Henle, F. V. (n.d.). *SQUARING AND NOT SQUARING ONE OR MORE PLANES*. https://hosted.math.rochester.edu/ojac/vol10/113.pdf. [Accessed 24 June 2021]

3. Ime.math.arizona.edu. (2021). Retrieved 24 July 2021, from http://ime.math.arizona.edu/ati/Math%20Projects/C1_MathFinal_Papenfus.pdf.

4. Jackson, Steve & Mauldin, R.. (1985). Survey Of The Steinhaus Tiling Problem. Bulletin of Symbolic Logic. 9. 10.2178/bsl/1058448676.

5. *Z. Moroń*. Squaring.net. (2021). Retrieved 24 July 2021, from http://www.squaring.net/history_theory/z_moron.html.