The Dollar Auction

Group 8-08

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1. Introduction and Rationale

The dollar auction is a non-zero sum sequential game designed by economist Martin Shubik to illustrate a paradox brought about by traditional rational choice theory in which players are compelled to make an ultimately irrational decision based completely on a sequence of apparently rational choices made throughout the game.

The setup involves an auctioneer who volunteers to auction off a dollar bill with the following rule: the bill goes to the winner; however, the second-highest bidder also loses the amount that they bid, making them the biggest loser in the auction. When either of the players bids, they bid 5 cents more than the highest current bid, starting with a bid of 5 cents. The game ends when a player passes, where the last player who bids is crowned the winner and given the dollar bill. The winner can even get a dollar for a mere 5 cents (the minimum bid).

In our research and investigations, we are working on a variant of this auction where the item on auction might be of different worth in value to each player.

In this game defined by (a, b, n, i), there are 2 players: A and B. An item is worth, in units to A and B respectively, a and b. There is a maximum number of moves, n. Each player can bid or pass at a move. A starts first with the bid incrementation, i (or he can pass). A and B will then bid sequentially for that item. Both players will pay at the end of the game, regardless of who wins. The person who loses pays their last bid to the auctioneer, while the winning bidder still pays to the auctioneer, and gets the item for his last bid. We assume that all players play rationally, that is, they try to lose as little money as possible.

1.1 Objectives

Through the analysis of this game, we hope to achieve three objectives. Firstly, we want to think of different ways the dollar auction could be altered. Secondly, to research the different variants of the dollar auction and how the changes affect the optimal outcome. Lastly, we want to gain insight about game theory and other relevant mathematical concepts.

1.2 Research Questions

Here are our research questions:

1. How would the bid incrementation affect the outcome of the game, given the values of the item to players A and B?
2. If there is a collusion between the two players, or one player and the auctioneer, how will this affect the outcome of the game?

3. How will the answers to the above questions change if there are more than two players?

1.3 Terminology

Term	Definition
Bid	Offering to pay an amount of money for an item.
Bid increments	The amount of money a player is allowed to increase the value of the bid.
Outcome	The amount of money paid by the players to the auctioneer at the end of the game, for the value of the item
Strategy	A rule or plan of action for playing a game.
Optimal Strategy	An optimal strategy is one that provides the best outcome for a player in a game.
Bid increments	The amount of money a player is allowed to increase the value of the bid.

1.4 Literature Review

Here is our literature review on the pieces of previous works we have come across.

1. *The Dollar Auction Game: A paradox in Non-cooperative Behaviour and Escalation (M. Shubik, 1971)*

The research paper gives us the basic idea of the Dollar Auction game. It provides the foundation and basic rules for us to build upon. The paper also introduces the concept of having players cooperate with each other for mutual benefits.

2. *Game theory through examples Erich Prisner (2014)*

_____The paper provides the solution for the basic form of the game and explores different variants of the game, providing some general ideas used in our research.

3. *Multilateral De-escalation in the Dollar by Auction Fredrik Ødegaard, Charles Z. Zheng (2017)*

_____The Research provides valuable insight on how to approach solving multiplayer forms of the Dollar Auction that we can use as a foundation for our own solutions.

2. Methodology

First, we researched online for relevant articles or works related to our research topic to gain insight and tried to formulate conclusions. Next, we simulated the game by playing it under the given rules. Then, we organised the findings and observations . Lastly, we attempted to prove the results mathematically with formulas and diagrams.

3. Results

We will be using a special result which we have came up with to solve the research questions:

lemma 1

By using only the outcomes of all games with a fixed bid incrementation, i, we can derive from them the outcomes of all possible games.

(Proof found in section 7.1)

Thus, in the solving of the research questions, we assume that the bid incrementation is fixed at 10 units for all examples before generalising the concluding statement.

3.1 Research Question 1

How would the bid incrementation affect the outcome of the game, given the values of the item to players A and B?

Below is the table of the amount of money in units each player has to pay for the different cases of the value of *i* with respect to *a* and *b*.

	$i \geq a$	$a > i \geq \frac{1}{2}a$	$i < \frac{1}{2}a$
$i > b$	A: 0 B: 0	*A: *i* B: 0	*A: *i* B: 0
$i = b$	A: 0 B: 0 *Or* A: 0 *B: 10	*A: *i* B: 0	*A: *i* B: 0
$b > i \geq \frac{1}{2}b$	A: 0 *B: *i*	*A: *i* B: 0	*A: *i* B: 0
$i < \frac{1}{2}a$	A: 0 *B: *i*	A: 0 *B: *i*	<u>*n* is even:</u> A: 0 *B: *i* <u>*n* is odd:</u> *A: *i* B: 0

*(An * before a player means that the player won the auction and got the item for his highest bid)*

The complete proof is found in section 7.2.

3.2 Research Question 2

If there is a collusion between the two players, or one player and the auctioneer, how will this affect the outcome of the game?

When 2 players form an alliance their main goal is to gain as much profit as possible as a whole.

Below is the table of the amount of money in units each player has to pay and the profit of the alliance for the different cases of the value of i with respect to a and b, when the 2 players formed an alliance.

	$i > a$	$i = a$	$i < a$
$i > b$	A: 0 B: 0 Profit: 0	*A: i B: 0 Profit: 0 *Or* A: 0 B: 0 Profit: 0	*A: i B: 0 Profit: $a - i$
$i = b$	A: 0 B: 0 Profit: 0 *Or* A: 0 *B: i Profit: 0	A: 0 B: 0 Profit: 0 *Or* A: 0 *B: 10 Profit: 0	*A: i B: 0 Profit: $a - i$

		Or *A: i B: 0 Profit: 0	
$i < b$	A: 0 *B: i Profit: 0	A: 0 *B: i Profit: $b - i$	$a = b$: *A: i B: 0 Profit: $a - i$ *Or* A: 0 *B: i Profit: $b - i$ $a > b$: *A: i B: 0 Profit: $a - i$ $a < b$: A: 0 *B: 10 Profit: $b - i$

*(An * before a player means that the player won the auction and got the item for his highest bid)*

When an auctioneer and a player team up, they want to make the other player pay as much as possible, regardless of who the item is sold to. Any other factor is irrelevant to them.

Below is the table of the amount of money in units each player has to pay and the profit of the alliance for the different cases of the value of *i* with respect to *a* and *b*, if player A and the auctioneer formed an alliance.

	$i \geq \frac{1}{2}a$	$i < \frac{1}{2}a$
$i > b$	*A: *i* B: 0 Profit: 0 *Or* A: 0 B: 0 Profit: 0	*A: *i* B: 0 Profit: 0 *Or* A: 0 B: 0 Profit: 0
$i = b$	A: 0 B: 0 Profit: 0 *Or* A: 0 *B: *i* Profit: *i*	A: 0 B: 0 Profit: 0 *Or* A: 0 *B: *i* Profit: *i*
$b > i \geq \frac{1}{2}b$	A: 0 *B: *i* Profit: *i*	A: 0 *B: *i* Profit: *i*
$i < \frac{1}{2}b$	<u>*n* is even:</u> A: $i \times (n-1)$	<u>*n* is even:</u> A: $i \times (n-1)$

	*B: $i \times n$ Profit: $i \times n$ <u>n is odd:</u> *A: $i \times n$ B: $i \times (n\text{-}1)$ Profit: $i \times (n\text{-}1)$ *Or* A: $i \times (n\text{-}2)$ *B: $i \times (n\text{-}1)$ Profit: $i\times (n\text{-}1)$	*B: $i \times n$ Profit: $i \times n$ <u>n is odd:</u> A: 0 B: 10 Profit: i

*(An * before a player means that the player won the auction and got the item for his highest bid)*

Below is the table of the amount of money in units each player has to pay and the profit of the alliance for the different cases of the value of i with respect to a and b, if player B and the auctioneer formed an alliance.

	$i \geq a$	$a > i > \frac{1}{2}a$	$i = \frac{1}{2}a$	$i < \frac{1}{2}a$
$i > \frac{1}{2}b$	A: 0 B: 0 Profit: 0 *Or* A: 0 *B: i	A: i *B: $2 \times i$ Profit: i	A: i *B: $2 \times i$ Profit: i	<u>n is even:</u> A: $10 \times (n\text{-}1)$ *B: $10 \times n$ Profit: $10 \times (n\text{-}1)$ *Or*

	Profit: 0			*A: 10 × (*n*-1) B: 10 × (*n*-2) Profit: 10 × (*n*-1) <u>*n* is odd:</u> *A: 10 × *n* B: 10 × (*n*-1) Profit: 10 × *n*
$i = \frac{1}{2}b$	A: 0 B: 0 Profit: 0 *Or* A: 0 *B: *i* Profit: 0	A: 0 B: 0 Profit: 0 *Or* A: 0 *B: *i* Profit: 0	A: *i* *B: 2×*i* Profit: *i*	<u>*n* is even:</u> A: 10 × (*n*-1) *B: 10 × *n* Profit: 10 × (*n*-1) *Or* *A: 10 × (*n*-1) B: 10 × (*n*-2) Profit: 10 × (*n*-1) <u>*n* is odd:</u> *A: 10 × *n* B: 10 × (*n*-1) Profit: 10 × *n*
$i < \frac{1}{2}b$	A: 0	A: 0	A: 0	<u>*n* is even:</u>

	B: 0 Profit: 0 *Or* A: 0 *B: i Profit: 0	B: 0 Profit: 0 *Or* A: 0 *B: i Profit: 0	B: 0 Profit: 0 *Or* A: 0 *B: i Profit: 0	A: 0 B: 0 Profit: 0 *Or* A: 0 *B: i Profit: 0 <u>n is odd:</u> *A: $10 \times n$ B: $10 \times (n\text{-}1)$ Profit: $10 \times n$

*(An * before a player means that the player won the auction and got the item for his highest bid)*

The complete proof is found in section 7.3.

3.3 Research Question 3

How will the answers to the above questions change if there are more than two players?

In an auction with k players defined by (v_1, v_2, v_3, ... , v_k, n), label the players as P_1, P_2, P_3, ... , P_k, and the item is worth v_1, v_2, v_3, ... , v_k to each of them respectively. The auction goes like this: Players take turns to bid, starting with the player with the lowest index in the auction to the highest index in the auction. Each player can bid or pass. When a player bids, he adds i units to the latest bid called so far. When a player passes, he leaves the auction and will no longer participate in it. The game ends when there is one player left in the auction or the limit of n rounds is reached, where the item is given to the last player who bid and everyone has to pay their highest bid regardless of whether they won the auction. However, if everyone passes in the first k moves, no one pays anything and the item is not given to anyone.

How will the game with k players turn out?

When there exists no $v_j > i$, no one pays anything and no one gets the item.

Otherwise, we let there be x such players $\{p_1, p_2, p_3, \dots , p_x\}$.

If there exists $v_j > k \times i$, the player $p_{((n-1) \text{ modulo } x) + 1}$ will pay 10 units and get the item while the other players will all pass. Otherwise, we let the set of players P_j with $\lfloor \frac{V_j}{10} \rfloor$ being the largest possible value among all j be $\{q_1, q_2, q_3, \dots , q_y\}$, the player $q_{((n-1) \text{ modulo } y) + 1}$ will get the item for 10 units while the others all pass.

What if all the players form an alliance?

A player P_j with v_j the largest possible value among all j will bid 10 units and get the item if $v_j > i$ and the alliance gains $(v_j - i)$ units as profit. Else, everyone passes on their first move and the alliance gains no profit.

What if a player P_j forms an alliance with the auctioneer?

If there exists no $v_j > i$, no one bids and the alliance does not gain any profit.

Otherwise, if P_j was the player the other players thought was going to win, the other players all pass on their first turn and the alliance does not gain any profit when $j = k$. If $j < k$, the game turns into a new one with k - j players after the first k players pass, the alliance will gain i units of profit or nothing depending on whether the k players have one with item worth > i.

If P_j was not the player the other players thought was going to win, say it was P_l, the alliance will gain a profit of ($i \times (n - 1)$) units (if $l < j$ and n is even OR $j < l$ and n is odd) or ($i \times n$) units (if $l < j$ and n is odd OR $j < z$ and n is even), because P_j and P_l will keep taking turns to bid while the other players pass on their first turn.

The complete proof is found in section 7.4.

4. Conclusion

For Research question 1, there is exactly 1 or no player who bids in each case. The player who bids gets the item for i units. If $a > i$, player A will be the one bidding if $b \leq 2 \times i$. Else, player B will bid if $a \leq 2 \times i$, except when $a > 2 \times i$ and $b > 2 \times i$ where the person bidding is decided by parity of n, and when $a \leq i$ and $b = i$, where what player B does is undetermined after player A bids.

For Research question 2, when the 2 players team up, a player with the highest possible value between them will bid, getting the most profit. If the auctioneer and a player team up, the alliance will gain profit of i if the item worth to the other player is > i and $\leq 2 \times i$ and the other player does not think the player in the alliance will bid in the next round. If the item worth to the other player is > $2 \times i$, the profit the alliance gains is decided by the parity of n.

For Research Question 3, in a game with k players, unless all have item worth < i where all players do not pay anything and do not get the item, there is exactly one player who bids i and gets the item while the others do not pay

anything. If there are x players with item worth $> k \times i$, the winner of the auction is the player with the
(*((n-1) modulo x)+1*)-th smallest index among them. If there does not exist players with item worth $> k \times i$, let there be y players in the form P_j such that $\lfloor \frac{Vj}{10} \rfloor$ being the largest possible value among all j, the winner of the auction is the player with the (*((n-1) modulo y) + 1*)-th smallest index among them.

If all the players team up, a player with the highest possible value among them will bid, getting the most profit. If the auctioneer teams up with a player, the alliance gets a profit of i if the player who teamed up is the one expected to win and there exists a player with item worth > i among those with a higher index than him and the profit depends on the parity of n if the player who teamed up is not the one expected to win. The alliance gains no profit besides these 2 cases.

5. Extensions

Here are some of the possible areas for further research in the future:

1. How would the strategy of the players change if the value of the item increases/decreases over time?

2. How would the strategy of the players change if bid incrementation increases in a sequence (eg., in multiples of 5; incrementation is 5 for 1st round, 10 for 2nd round, 15 for 3rd round etc.)?

6. References

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7. Appendix

7.1 Proof of *lemma 1*

Let c represent the bid incrementation in each game.

Say that we are considering the case:

- $c = w$
- $a = x$
- $b = y$
- $n = z$

Now consider the new case:

- $c = i = w \times (i / w)$
- $a = x \times (i / w)$
- $b = y \times (i / w)$
- $n = z$

The ratio for the different values (i.e. bid incrementation, value of object to A, value of object to B, difference between bid incrementation and value of object to A, difference between bid incrementation and value of object to B, difference of value of object to A and value of object to B) is the same in the old case and the new case.

We realise that in the 2 cases, players A and B will be carrying out the same set of moves, when they play optimally. The outcome of the new case can also be determined, knowing that the ratio of final payment of A, final payment of B, amount paid to auctioneer in total from the old case to the new case are all $w : i$.

Thus, we can find the outcome of any case by only considering the cases where the bid incrementation is a certain value, i.

7.2 Proof of Research Question 1

We start by considering the value of the item to player B.

Case 1: $b < 20$

Let the highest bid that player B has placed so far be x units. If player B bids on his next move, he would have to pay $(x + 20)$ units or more and get or lose the item, thus the lowest value he would have given to the auctioneer at the end of the game is $(x + 20 - b)$ units.

However, $x + (20 - b) > x + 0 = x$

Thus player B will not bid but pass on any move of his in order to spend the least amount of units, meaning that player B will pass on his first move.

Now, consider the item worth to player A.

Case 1.1: $a < 10$

Player A will not bid on the first move as he would rather spend 0 units and not get the item than spending 10 units to get the item, as $10 - a > 0$. Player B will bid if $b > 10$, pass if $b < 10$ and it is undetermined what he will do if $b = 10$.

Case 1.2: $a = 10$

Player A will pass in the first move as both choices are worth the same in value to him and it is the safer choice in case player B decides to bid after he does. Player B will bid if $b > 10$, pass if $b < 10$ and it is undetermined what he will do if $b = 10$.

Case 1.3: $a > 10$

Player A will bid on the first move, as he knows player B will pass in the next, and get the item for 10 units.

Case 2: $b = 20$

Consider the item worth to player A.

Case 2.1: $a < 10$

Player A will pass on the first move, then player B will bid in the next and get the item for 10 units.

Case 2.2: $a = 10$

Player A will pass on the first move as he cannot determine whether player B will bid or pass in the next, considering that passing will earn him 0 units while bidding will provide a chance where he bids 10 units or more for nothing. Playing optimally, he will pass in the first move to stay safe. Player B will bid in the next and get the item for 10 units.

Case 2.3: $10 < a < 20$

If player A bids on the first move, player B will pass on the second move as it is the safer choice in case player A decides to bid on the third move for some unforeseen reason, allowing player A to get the item for 10 units.

Case 2.4: $a = 20$

Player A will bid 10 units on the first move. Following that, player B will be unsure whether player A will bid in the third turn, thus player B will pass to play safe and prevent any chance of losing money as bidding will not earn him more money than passing anyways. Thus player A gets the item for 10 units.

Case 2.5: $a > 20$

He will bid 10 units on the first move. Then, player B will pass to play safe, as otherwise player A might bid on the third move and cause player B to pay more money. Player A gets the item for 10 units.

Case 3: $b > 20$

Once again consider the item worth to player A.

Case 3.1: $a < 10$

He will not bid in the first move and thus player B will bid and get the item for 10 units.

Case 3.2: $a = 10$

If player A bid on his first move, player B will bid on the next as he knows player A will pass after that, causing player A to lose 10 units for nothing. Thus he will not bid on the first move, following which player B bids and gets the item for 10 units.

Case 3.3: $10 < a < 20$

Following a similar logic as Case 3.2, player A will skip the first move and let player B get the item for 10 units.

Case 3.4: $a = 20$

If player A bids on his first turn, following the same logic of player B in Case 2.4, he will pass on his next turn, knowing this, player B will bid on his first turn. This will cause player A to lose 10 units for nothing. Thus, player A will pass on his first turn, allowing player B to bid 10 units and get the item.

Case 3.5: $a > 20$

Since getting the item by bidding 20 units more is always better than passing in terms of value, both players will bid continuously, eventually reaching the limit of n rounds. If the last turn goes to player B (n is even), player A would not have bid but instead passed in the second-to-last move, in order to lose less money. Then, knowing that, player B in the third-to-last move would have bid. Thus, player A in the fourth-to-last move would not have bid. Following this pattern, player A would keep passing. Eventually in the second move, B would have bid 20 units, since player A would have passed in the third move, he would also pass the first move to lose less money. This meant that player B bid 10 units on the second move after player A's pass and got the item. If the last turn goes to player A (n is odd), following the same logic, player B would always pass. Thus, player B would have passed on the second move and player A would bid on the first move and get the item for 10 units.

7.3 Proof of Research Question 2

When the two players work with each other:

If the two players are trying to work together, they want to benefit the most in terms of value as a whole.

Case 4: $a < 10$

Case 4.1: $b < 10$

Both of them would not want the item even for 10 units, the lowest they can possibly get it for, thus they will both pass and no one gets the item.

Case 4.2: $b = 10$

Player A will pass in the first turn as he would not want the item, it is then undetermined whether player B will bid or pass.

Case 4.3: $b > 10$

Player A will pass, to allow player B to get the item for 10 units, which he does not get anyways, together they make a profit of (b - 10) units.

Case 5: $a = 10$

Case 5.1: $b < 10$

It is undetermined whether player A will bid or pass in the first move and player B will pass in the next.

Case 5.2: $b = 10$

It is undetermined whether player A will bid or pass in the first move. If he bids, player B passes in the next. If he passes in the first move, it is again undetermined whether player B will bid or pass in the next.

Case 5.3: $b > 10$

Player A will pass, allowing player B to get the item for 10 units in the next round and gaining a maximum profit of (b - 10) units in the next round.

Case 6: $a > 10$

Case 6.1: $b < 10$

Player A will bid and then player B will pass, gaining a profit of (a - 10) units.

Case 6.2: $b = 10$

Player A will bid and then player B will pass, gaining a profit of (a - 10) units.

Case 6.3: $b > 10$

Case 6.3.1: $a = b$

Player A can either bid or pass, then player B in the next round will do the opposite move, together gaining a profit of (a - 10) = (b - 10) units.

Case 6.3.2: $a > b$

Player A will bid and then player B will pass, gaining a profit of (a - 10) units as (a - 10) > (b - 10).

Case 6.3.3: $a < b$

Player A will pass and then player B will bid, gaining a profit of (b - 10) units as (b - 10) > (a - 10).

When a player and the auctioneer work together:

The amount spent by the player and whether the item is sold or who it is sold to are ignored, the main goal is to obtain the most amount of money possible from the other player, who does not know the alliance between the player and the auctioneer.

Case 7: Player A and the auctioneer formed an alliance

Case 7.1: $b < 10$

Player B will always pass, not letting the alliance gain any amount of money.

Case 7.2: $b = 10$

Player A will pass in the first round then it will be undetermined whether player B will bid or pass, the alliance will gain 0 and 10 units respectively.

Case 7.3: $10 < b < 20$

Player A will pass, then player B will bid, letting the alliance gain a profit of 10 units.

Case 7.4: $b = 20$

Case 7.4.1: $a < 20$

If player A bids first, player B thinks that player A will pass on the next move, he can either bid or pass as both moves are worth the same to him, it will be uncertain whether the alliance can gain profit. However, if player A passes first, player B will bid next to gain 10 units for himself and the alliance will surely gain 10 units. Thus player A will pass first.

Case 7.4.2: $a = 20$

The outcome is similar to that of Case 7.4.1, player A will pass first, the alliance will gain 10 units.

Case 7.4.3: $a > 20$

If player A bids first, player B will pass as he thinks player A will bid in the next round, thus the alliance will gain 0 units. If player A passes, player B will bid and the alliance will gain 10 units.

Case 7.5: $b > 20$

Case 7.5.1: $a < 20$

Player B will keep thinking that player A will pass in the next move thus he will keep bidding, giving the alliance a profit of

($10 \times n$) units (n is even) or ($10 \times (n - 1)$) units (n is odd).

Case 7.5.2: $a = 20$

Similar to Case 3.4, player B will keep thinking that player A will pass in the next move thus he will keep bidding, giving the alliance a profit of

, giving the alliance a profit of

($10 \times n$) units (n is even) or ($10 \times (n - 1)$) units (n is odd).

Case 7.5.3: $a > 20$

If n is even and player A bids on the first turn, it turns into a game with A limit of (n - 1) moves with B starting first. Similar to the logic in Case 3.4, player B will keep thinking that player A will pass in the next move thus he will keep bidding, giving the alliance a profit of ($10 \times n$) units. If n is odd and player A bids on his first turn, player B will pass on the second turn as he will deduce that player A will get the item in the end, similar to Case 3.5, thus player A will pass on the first turn, allowing player B to get the item for 10 units and the alliance will gain 10 units of profit.

Case 8: Player B and the auctioneer formed an alliance

Case 8.1: $a < 10$

Player A will always pass, not letting the alliance gain any amount of money.

Case 8.2: $a = 10$

Case 8.2.1: $b < 20$

Player A will pass as both choices are worth the same to him in value and it is the safer choice in case player B decides to bid for some unforeseen reason.

Case 8.2.2: $b = 20$

Similar to Case 8.2.1, player A will pass first, the alliance will not gain any profit.

Case 8.2.3: $b > 20$

Similar to Case 8.2.1, player A will pass first, the alliance will not gain any profit.

Case 8.3: $10 < a < 20$

Case 8.3.1: $b < 20$

Player A will bid first, thinking that player B will pass on the next move, then player B will bid due to the alliance and player A passes after that to prevent losing more money. The alliance gains 10 units.

Case 8.3.2: $b = 20$

Similar to Case 2.3, player A will pass on the first move and the alliance will not gain any profit.

Case 8.3.3: $b > 20$

Similar to Case 3.3, player A will pass on the first move and the alliance will not gain any profit.

Case 8.4: *a* = 20

Case 8.4.1: $b < 20$

Similar to Case 1.3, player A will think that player B will keep passing and bid on the first move, then player B will bid and player A will pass next to prevent losing more money. The alliance gains a profit of 10 units.

Case 8.4.2: $b = 20$

Similar to Case 2.4, player A will think that player B will pass on the second move and bid on the first move, then player B will bid and player A will pass next to prevent losing more money. The alliance gains a profit of 10 units.

Case 8.4.3: $b > 20$,

Similar to Case 3.4, player A will pass on his first turn, the alliance will not gain any profit.

Case 8.5: *a* > 20

Case 8.5.1: $b < 20$

Player A will keep thinking that player B will pass on the next move, thus he will keep bidding. The alliance will gain
$(10 \times (n - 1))$ units (n is even) or $(10 \times n)$ units (n is odd).

Case 8.5.2: $b = 20$

Similar to Case 2.5, player A will keep thinking that player B will pass on the next move, thus he will keep bidding. The alliance will gain $(10 \times (n - 1))$ units (n is even) or $(10 \times n)$ units (n is odd).

Case 8.5.3: b > 20

If n is even, player A will pass on his first turn as he will deduce that player B will get the item in the end, similar to Case 3.5, thus he will pass on the first turn and the alliance will not gain any profit. If n is odd, player A will keep bidding and so will player B due to the alliance and the alliance will gain a profit of ($10 \times n$) units.

7.4 Proof of Research Question 3

How will an auction with k players turn out:

Let us define a cycle to be the series of moves where each person remaining in the auction makes a move once, starting with the player with the lowest index in the auction to the highest index.

In the first cycle, all players P_i with $v_i < 10$ will pass and leave the auction as it is always more worth to pass than bid.

Case 9: $v_1, v_2, v_3, \ldots, v_k \leq 10$

All players pass in the first cycle and no one gets the item or a single player gets the item for 10 units while the others do not pay any money.

Case 10: Not all $v_1, v_2, v_3, \ldots, v_k \leq 10$

Those players P_i with $v_i < 10$ will all pass in the first round and leave the auction , as passing will be more beneficial than bidding at any move. Now we consider whether there exists i such that $v_i > k \times i$.

Case 10.1: There exists i such that $v_i > k \times i$

Let there be x such players. All players P_i with $v_i \leq k \times i$ will pass in the first cycle, they know that the players P_i with $v_i > k \times i$ will continue bidding after the first cycle, thus it is more beneficial to pass in the first round in order to prevent losing more money. Let $S = \{p_1, p_2, p_3, \ldots, p_x\}$ be the set of players P_i who remain in the auction, from the smallest P index to the largest P index. Similar to Case 3.5, the rest of the players will bid until the limit of n rounds. The other players who bid in the second-to-last round, third-to-last round, …, x-to-last round will pass as they will deduce that the player bidding in the last move (let this person be p_j where $j = n$ modulo x if n modulo $x > 0$ and otherwise $j = x$) will get the item. Then, in the (x

- *1*)-to-last round, the player will bid in that move as he knows the other players will pass. Following this pattern, all the x players except for P_j will pass, even in the first cycle. Thus, everyone except P_j will pass in the first round, letting P_j get the item for 10 units.

Case 10.2: There does not exist i such that $v_i > k \times i$

Let $S = \{p_1, p_2, p_3, \ldots, p_x\}$ be the set of players P_i with $\lfloor \frac{Vi}{10} \rfloor$ equal to the largest possible value among all such expressions of $v_1, v_2, v_3, \ldots, v_k$, from the smallest P index to the largest P index. All players except for the players in S will pass in the first round, as if they bid, they would need to bid more than what the item is worth to them in order to get the item, as the players in S will have no problem bidding up to ($\lfloor \frac{Vi}{10} \rfloor \times 10$) units. Similar to Case 3.5, the players in S will bid until the limit of n rounds. The other players who bid in the second-to-last round, third-to-last round, …, x-to-last round will pass as they will deduce that the player bidding in the last move (let this person be p_j where $j = n$ modulo x if n modulo $x > 0$ and otherwise $j = x$) will get the item. Then, in the (x-1)-to-last round, the player will bid in that move as he knows the other players will pass. Following this pattern, all the x players except for p_j will pass, even in the first cycle. Thus, everyone except p_j will pass in the first round, letting p_j get the item for 10 units.

Case 11: All players form an alliance

Case 11.1: All $v_1, v_2, v_3, \ldots, v_k < 10$

The players will all pass in the first cycle.

Case 11.1: Not all $v_1, v_2, v_3, \ldots, v_k < 10$

A player P_i with v_i equal to the largest possible value among all $v_1, v_2, v_3, \ldots,$ v_k bids in the first cycle, while the others all pass, allowing the alliance to gain a maximum of (v_i - 10) units.

Case 12: The auctioneer forms an alliance with a player

Let the player be P_z.

Case 12.1: All players have $v_1, v_2, v_3, \ldots, v_k < 10$

The alliance will be unable to gain any profit.

Case 12.2: Not all players have $v_1, v_2, v_3, \ldots, v_k < 10$

Note that in Case 10, the original game with no alliances, there is a pattern throughout all its sub-cases where a single player bids in the first cycle while the others all pass.

Case 12.2.1: The player who formed an alliance with the auctioneer is the player whom other players deduced to win the auction:

The players who have a smaller index than z will all pass, thinking that P_z will bid.

Case 12.2.1.1: $z = k$

The alliance will not gain any profit.

Case 12.2.1.1: $z < k$

P_z will pass on his move in the first cycle, the auction will turn into a new one with $P_{z+1}, \ldots, P_k$ as players and the alliance will gain 10 units of profit if some of P_i among them has $v_i > 10$ or 0 units otherwise.

Case 12.2.1: The player who formed an alliance with the auctioneer is not player whom other players deduced to win the auction:

The player who was deduced to win the auction, say P_y, and P_z will continue to bid until the limit of n rounds. The alliance will gain

$(10 \times (n - 1))$ units ($y < z$ and n is even OR $z < y$ and n is odd) or $(10 \times n)$ units ($y < z$ and n is odd OR $z < y$ and n is even).