Hwa Chong Institution (High School)

# Modifying the Pythagorean Theorem

Group 8–06

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## 1 Abstract

In this project, we used a computer algorithm to find solutions satisfying $a^2 + b^2 + c^2 = d^2$ when $a$, $b$, $c$, and $d$ are positive integer values, and use these solutions to find a formula which could generate more such Pythagorean quadruples.

## 2 Introduction

The Pythagorean Theorem describes the relationship between two sides of a right-angled triangle and its hypotenuse (ie. the longest side of the right-angled triangle, *Fig. 1*), where $a^2 + b^2 = c^2$, *c* representing the length of the hypotenuse, and *a* and *b* representing the lengths of the two other sides of the triangle.

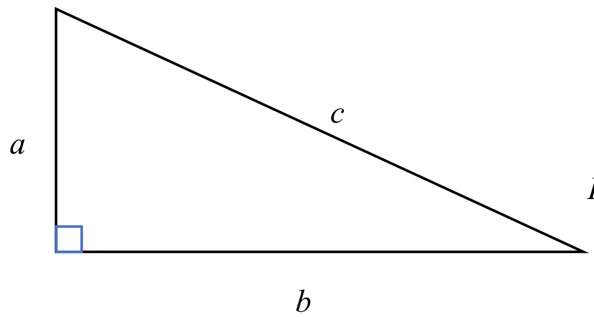


*Fig. 1*, a typical right-angled triangle.

However, this formula can be expanded on, including the addition of more terms to the original formula. In our modified formula, we have chosen to include one more term, $d^2$, into the original formula. The modified formula is as such:

$$a^2 + b^2 + c^2 = d^2$$

This project aims to investigate more about this formula, by:

- coming up with a two-dimensional geometric figure which satisfies the new, modified formula, and
- coming up with another definite formula which can generate more solutions for the formula above, namely $a^2 + b^2 + c^2 = d^2$.

## 2.1 Rationale

This project aims to explore more possibilities pertaining to the Pythagorean Theorem, including the addition and inclusion of more two-dimensional shapes and/or figures which could satisfy the modified formula, therefore expanding the scope of the Pythagorean Theorem outside of the right-angled triangle.

In addition, it also aims to find more sets of Pythagorean quadruples (ie. sets of four positive integers which could satisfy $a^2 + b^2 + c^2 = d^2$). We believe that finding a formula which could generate more such sets of Pythagorean quadruples would be a much more efficient and error-proof approach than using manual trial and error to do so.

## 2.2 Research Aims and Problems

a. To investigate primitive Pythagorean quadruples
b. To develop a brute-force computer algorithm which can generate such Pythagorean quadruples
c. To come up with a two-dimensional shape to satisfy the Pythagorean quadruples, in turn satisfying the formula $a^2 + b^2 + c^2 = d^2$
d. To come up with a formula which can generate Pythagorean quadruples without using brute force

# 3 Literature Review

Before embarking on investigating $a^2 + b^2 + c^2 = d^2$, we needed to do some research on whether this was in the first place possible.

In Elkies, N. D.'s "On $a^4 + b^4 + c^4 = d^4$", Euler conjectured in 1769 that the Diophantine equation $a^4 + b^4 + c^4 = d^4$, has no solution in positive integers. However, Elkies' team had come up with solutions using a computer algorithm. Citing the study, we therefore decided that there is a possibility of $a^2 + b^2 + c^2 = d^2$ having infinite solutions when $a$, $b$, $c$, and $d$ are positive integers. We also hypothesised that as both equations were in the same format, there would be infinite solutions satisfying $a^2 + b^2 + c^2 = d^2$.

# 4 Study & Methodology

## 4.1 Brute-Force Computer Algorithm

To obtain a set of primitive solutions for the equation $a^2 + b^2 + c^2 = d^2$, a computer algorithm was chosen to generate a **finite** number of solutions and present them in a human-readable format. The Python programming language was chosen, due to its capability to work with large numbers very quickly.

The computer algorithm loops through the different integer values of *c* from 1 to 100 inclusive, then determining if the values for $a^2 + b^2 + c^2$ and $d^2$ are equal. If so, the program will write that set of solutions for *a, b*, *c*, and *d* to an external text file. The same incrementation process applies for the values of *a* and *b*, the only difference being *b* incrementing 100× slower than *c*, and *a* incrementing 100× slower than *b*. The code can be found in **Appendix A**.

Below is a flowchart detailing the process (*Fig. 2*).

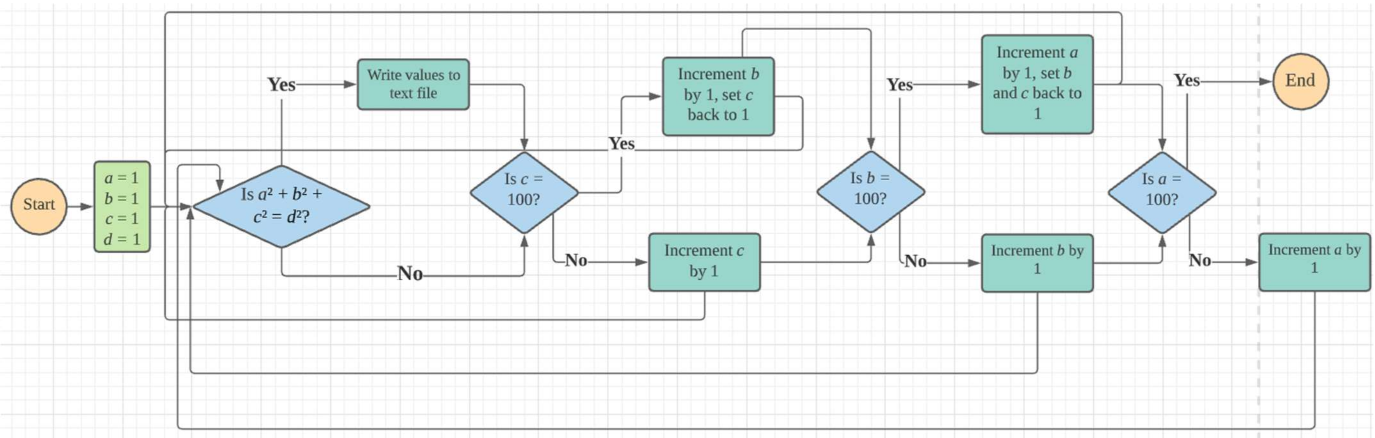


*Fig. 2*, a flowchart of the computer algorithm.

The first seven and last seven solutions (where $a, b, c, d \in \mathbb{Z}^+$ and $a, b, c \leq 100$) are presented as follows (*Fig. 3*):

| Solutions to $a^2 + b^2 + c^2 = d^2$, where $a, b, c, d \in \mathbb{Z}^+$ and $a, b, c \leq 100$ | | | |
|---|---|---|---|
| *a* | *b* | *c* | *d* |
| 1 | 2 | 2 | 3 |
| 1 | 4 | 8 | 9 |
| 1 | 6 | 18 | 19 |
| 1 | 8 | 32 | 33 |
| 1 | 10 | 50 | 51 |

| | | | |
|---|---|---|---|
| 1 | 12 | 12 | 17 |
| 1 | 12 | 72 | 73 |
| ⋮ | | | |
| 80 | 84 | 87 | 145 |
| 80 | 90 | 96 | 154 |
| 84 | 84 | 98 | 154 |
| 85 | 100 | 100 | 165 |
| 86 | 91 | 98 | 159 |
| 88 | 88 | 89 | 153 |
| 92 | 96 | 96 | 164 |

*Fig. 3*, the first and last seven solutions to $a^2 + b^2 + c^2 = d^2$ and $a, b, c \leq 100$.

The set of solutions generated by the computer algorithm provides a baseline for other areas of the Pythagorean quadruple investigated in this project, such as testing and experimenting with geometric figures.

## 4.2 Geometric Figure

We decided to create a 2-dimensional figure to visually represent $a^2 + b^2 + c^2 = d^2$. The figure we had come up with was a trapezium, composed of two right-angled triangles (*Fig. 4*).

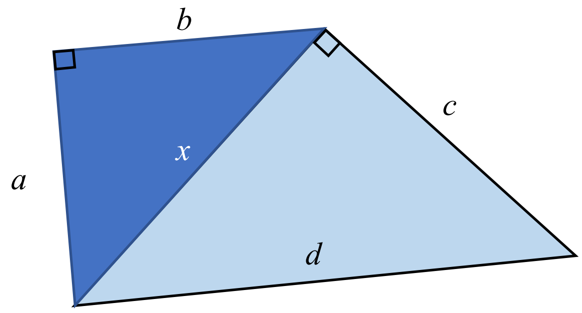


*Fig. 4*, a figure illustrating $a^2 + b^2 + c^2 = d^2$.

$$\because a^2 + b^2 = x^2, x^2 + c^2 = d^2 \qquad \therefore a^2 + b^2 + c^2 = d^2$$

Testing this figure against algorithm-generated solutions would prove that this figure represents $a^2 + b^2 + c^2 = d^2$. This shape would also form the basis for our formula, described in the next section.

## 4.3 A Definite Formula

As explained above, there are two occurrences of the Pythagorean theorem, in $a^2 + b^2 = x^2$ and $c^2 + x^2 = d^2$, which form the basis for our formula.

We first start with the formula for finding Pythagorean triples ($a, b,$ and $c$):

$$\text{Let } k, l \in \mathbb{Z}^+ \text{ and } k \neq l.$$

$$a = k^2 - l^2$$

$$b = 2kl$$

$$c = k^2 + l^2$$

With the new equation, $a^2 + b^2 + c^2 = d^2$, we can use the previous formula, via substitution, and the addition of two more variables:

Let $k, l, m, n \in \mathbb{Z}^+$ and $k \neq l \neq m \neq n$, $k > l$ and $m > n$

$a^2 + b^2 = (m^2 - n^2)^2$, $c^2 = (2mn)^2$, $d^2 = (m^2 + n^2)^2$.

$\because a^2 + b^2 = (m^2 - n^2)^2$ and $a^2 + b^2 = c^2$

$\therefore c^2 = (k^2 + l^2)^2 = (m^2 - n^2)^2$

$$\therefore \sqrt{(k^2 + l^2)^2} = \sqrt{(m^2 - n^2)^2} \Rightarrow k^2 + l^2 = m^2 - n^2$$

We can test this formula by substituting algorithm-generated values into the variables:

$$k = 2, l = 1, m = 3, n = 2$$

$$\therefore a = 3, b = 4, c = 12, d = 13$$

We have now acquired two sets of variables; the second set being derived from the first. There are multiple possibilities that were tested in the process of deriving the formula, but the findings showed that only performing addition or subtraction on variables of the second power, and multiplication of variables of the first power could enable the two sets of variables to be correlated in the same way as just demonstrated.

The final derived formula is as such:

$$a = k^2 - l^2, b = 2kl, c = 2mn, d = m^2 + n^2$$

## 5 Real-Life Applications

It is a gross misunderstanding that the Pythagorean quadruples, even the Pythagorean Theorem itself, has applications only in mathematics and science. Although this project has taken its roots in geometry, basic number theory, and basic algebra, it has many other real-world applications as well.

One of the applications of the Pythagorean quadruples is in the field of game engineering. The developers of the game engine *Quake III* used the Pythagorean Theorem in its algorithm for vector normalisation, which was required for the implementation of physics into their game engine (Lomont, 2003). For the game engine to normalise vectors, it had to take the reciprocal of the length of the vector, which can be found with $\sqrt{x^2 + y^2 + z^2}$, shown in *Fig. 5* below.

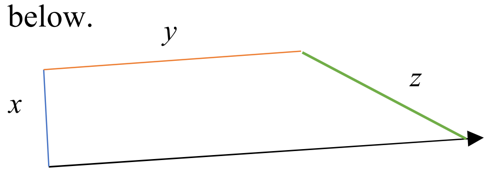


*Fig. 5*, length of a vector.

## 6 Conclusion & Further Investigations

In this project, we have researched on the formula $a^2 + b^2 + c^2 = d^2$ and through the course of the project, have come up with multiple aspects of solving this, such as the geometrical side of the equation – coming up with a geometric figure that could satisfy the equation, and a solution-generating formula that can come up with more solutions to satisfy the equation. We have also developed an algorithm to prove and solidify our points and how this can be used in real life other than the fields of mathematics and science. Further investigations for this project are possible, such as researching on different indices, or expanding the scope to surds, complex numbers, irrational numbers etc. All in all, we have achieved our goals of finding a formula and solutions to the equation, and this project has a lot of potential for further research and comparison.

# 7 Appendices

## 7.1 Appendix A

```
import math as m

a, b, c = 1, 1, 1
terms_list = []
lists_of_terms_lists = []
max_no = 100
total = 0
counter = 0

while a <= max_no:
    while b <= max_no:
        while c <= max_no:
            total = a ** 2 + b ** 2 + c ** 2
            d = m.sqrt(total)
            check = total % total ** 0.5 == 0

            if check is True:
                if a > b or b > c or a > c:
                    print("Combination already found: " + str(a) + "   " +
str(b) + "   "
                          + str(c) + "   " + str(d))
                else:
                    print("NEW combination found: " + str(a) + "   " + str(b)
+ "   " + str(c)
                          + "   " + str(d))
                    terms_list = [a, b, c, int(d)]
                    lists_of_terms_lists.append(terms_list)
                    counter += 1
            else:
                print("No combination found: " + str(a) + "   " +
                      str(b) + "   " + str(c) + "   " + str(d))
            c += 1
        b += 1
        c = 1
    a += 1
    b = 1
    c = 1

f = open("diophantine_equations1.txt", 'w')
f.write("Number of unique combinations: " + str(counter) + "\n")

for i in range(len(lists_of_terms_lists)):
    f.write(str(lists_of_terms_lists[i]))
```

```
        f.write("\n")
        i += 1
    f.close()
```

## 7.2 Appendix B - Figures Used

| S/N | Description | Page Number | Credits |
|---|---|---|---|
| 1 | *A typical right-angled triangle.* | 2 | Word, original |
| 2 | *A flowchart of the computer algorithm.* | 4 | Lucidchart, original |
| 3 | *The first and last seven solutions to* $a^2 + b^2 + c^2 = d^2$ *and* $a, b, c \leq 100$. | 4 – 5 | Word, original |
| 4 | *A figure illustrating* $a^2 + b^2 + c^2 = d^2$. | 5 | Word, original |
| 5 | *Length of a vector.* | 7 | Word, original |

# 8 References


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