Mathematical Project

How to Win in Japanese Mahjong

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Content

Introduction

Mahjong is a tile-based game originated from the Qing dynasty in China and has spread around the world since the 20^{th} century. It has multiple variants in different regions and rules also vary. Japanese mahjong, also known as riichi mahjong, is a variation with unique rules such as riichi, yaku, and the use of dora, which make it more emulous. Our group aims to research the mathematical principles behind Japanese mahjong and develop a mathematical method to win in Japanese Mahjong so that more people can understand the charm of it.

Objectives

1. We aim to find a way to determine whether we should attack or defend by proving and applying the research formula.

2.We decide to investigate different winning situations based on classical probability and expectation models.

3. Based on different situations, we aim to determine which kinds of combinations of tiles have a higher probability to win the game.

Basic Rules of Japanese mahjong

Japanese mahjong is usually played with 136 tiles. The tiles are mixed and then arranged into four walls that are each two stacks high and 17 tiles wide. 26 of the stacks are used to build the players' starting hands, 7 stacks are used to form a dead wall, and the remaining 35 stacks form the playing wall.

There are 34 different kinds of tiles, with four of each kind. Just like standard mahjong, there are three suits of tiles, pin (circles), sō (bamboo) and wan (characters), and unranked honor tiles. Honor tiles are further divided between wind tiles and dragon tiles. Some rules may have red number five tiles which work as dora that earn more han value. The flower and season tiles are omitted. Names for suit tiles follow the pattern of [number] + [suit], the numbers being Japanese pronunciations of the corresponding Chinese words.

There is a distinction between winning from the wall and winning from a discard. When going out, players call out "tsumo" (self-drawn) or "ron" (picking up a discard). In the case of tsumo, the other three players share the responsibility of paying out points, according to the scoring table. For ron, the player who discarded the tile pays all of the points.

iipin	ryanpin	sanpin	sūpin	ūpin	rōpin	chiipin	pāpin	chūpin

Pin

iisō	ryansō	sansō	sūsō	ūsō	rōsō	chiisō	pāsō	chūsō

So

iiwan	ryanwan	sanwan	sūwan	ūwan	rōwan	chiiwan	pāwan	chūwan

Man

ton (East)	nan (South)	shā (West)	pei (North)

Wind tiles

haku	hatsu	chun

Dragon tiles

Literature Review

1.Rules of Japanese Mahjong
By European Mahjong Association (2016 edition)

We apply the rules of Japanese mahjong in this document.
There are two unique rules of Japanese Mahjong. Yaku and Riichi.
Yaku is the main reason that makes Japanese Mahjong more competitive than other forms of Mahjong. This is because others do not consider Yaku at all, as long as a player gets four melds, he will win the game. More restrictions and limitations will result in the larger role of strategy a player takes to win the game.
Riichi is an important situation we need to consider in our project, and we will use a mathematical model to determine whether to riichi or not and when to riichi in a game.

2. 麻雀戦術書　ビギナーズラック(The Theory of Mahjong)

This document mainly describes the theories and strategies of controlling the hands, the best timing to Tenpai and how to maximise tile efficiency.
We get some key factors that determine the strategy of choosing offence or defense from this document:

(1) Possibility of winning by using the tiles given

(2) Your winning points

(3) Probability of letting your opponents win when you discard a certain tile

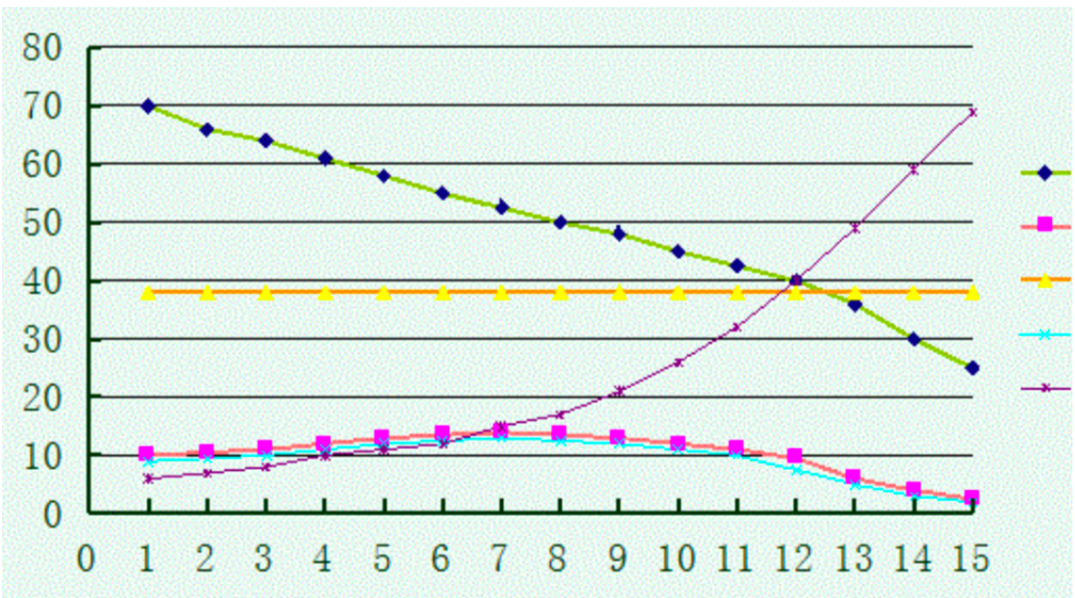
(4) Prediction of your opponents' winning points

Besides, we get our idea about the pros and cons for riichi, tile efficiency, and how to control hands from this book. These factors are constantly influenced by lots of variables during the entire game.

3. おしえて！科学する麻雀 (Tell me! Mahjong to science)

By とつげき東北

This document discusses the expected income of riichi at different rounds in the game, which helps us determine whether to riichi or not according to different situations. We also get our research formula about the judgement of choosing to attack or defend and the effect of riichi on the probability of winning the game.



(The graph of the percentage against the turns that we choose to riichi)

The green line is your own winning rate,
The pink line is the winning rate of your opponents,
The orange line is rate of winning by own draw,
The blue line is the winning rate of your opponents by their own draws
and the purple line is the rate of tie in the game

Benefits of riichi

1.we can gain a yaku worth 1 han from riichi. This is especially important for hands where we start with a number of dora tiles and we do not have yaku.

2.It can be used as a method to defend. As shown in the graph, the pink line represents the rate of letting other people win when we riichi. We can see that after the 7th rounds, riichi can decrease the winning rate of other players.

Disadvantages of riichi

Riichi is a sign that alerts other players that you are in tenpai. They may choose to defend (not let you win) after you choose to riichi. This can be shown from the green line in the graph. The green line represents your own winning rates. We can see it decreases sharply throughout the game when we choose to riichi.

Research Formula

by おしえて！ 科学する麻雀 (Tell me! Mahjong to science)

G(the income we expect for offense):

$$G(o) = (1 - p)\{wW - h(H + r) - t(T + r)\} - pH$$

G(the income we expect for defense):

$$G(d) = -oT$$

Hence, the discriminant will be

$$D = G(o) - G(d) = (1 - p)\{wW - h(H + r) - t(T + r)\} - pH + oT$$

Terminologies & Notations

T: The average points we lose when other players win by their own draw

t: The rate of other players win by own draw after we riichi (not the first one to riichi)

W: The income we get when we win including riichi rod

w: The rate that we can win when we discard the tile which is safe to be discarded

r: If we are at riichi state, r equals to 1000; if not, r equals to 0

h: The rate of letting others win when we discard the tile which is a ‘safe discard’ (The safe here is simply low possibility of letting others win the game, a ‘safe discard’ is not a completely safe.)

H: The average points we lose when other players win after we discard the tile

o: The rate that other players win by their own draw

p: The rate that other players would win when we discard the tile

Proof

$$\mathbf{D} = \mathbf{G}(\mathbf{o}) - \mathbf{G}(\mathbf{d})$$

D, the discriminant between G(o) and G(d), shows how much is G(o) greater (or less) than G(d).

If D>0, it means that the player will have more income if he plays offense.

If D<0, it means that the player will have more income if he plays defense.

(1) G (o), which is short for G (the income we expect for offense) is the expected points a player would get if he goes for offense.

According to probability theory, E(X) $= \sum_{k=1}^{\infty} x_k p_k$, in which E(x) is the expect value of variable x, p_k is the probability an incident k to occur and x_k is the income for incident k

Thus,

G(o)

=E(x)

=x (win by offense) * p (win by offense) + x (lose by offense) * p (lose by offense)

Clearly, we have:

$$\begin{cases} \text{p (win by offense)} = (1-p) \\ \text{x (lose by offense)} = -H \\ \text{p (lose by offense)} = p \end{cases}$$

Thus, G(o) = (1-p) * x (win by offense) – pH

For x (win by offense): E(X) $= \sum_{k=1}^{\infty} x_k p_k$

x (win by offense)

=x (win by safe discard) * p (win by safe discard) + x (lose by discarding a tile) * p(lose by discarding a tile) + x(lose by other players tsumo) * p(lose by other players tsumo)

Clearly, we have:

$$\begin{cases} x\ (\text{win by safe discard}) = W \\ p\ (\text{win by safe discard}) = w \\ x\ (\text{lose by discarding a tile}) = -(H+r) \\ p\ (\text{lose by discarding a tile}) = h \\ x\ (\text{lose by other players tsumo}) = -(T+r) \\ p\ (\text{lose by other players tsumo}) = t \end{cases}$$

Thus, x (win by offense) = wW – h(H+r) -t(T+r)

*Note: tsumo means winning by self-draw

Thus, $G(o) = (1-p)\{wW - h(H+r) - t(T+r)\} - pH$

(2) G(d) which is short for G (the income we expect for defense) is the expected points a player would get if he goes for defense.

G(d) = x (other players tsumo) * p (other players tsumo)

$$\begin{cases} x\ (\text{other players tsumo}) = -T \\ p\ (\text{other players tsumo}) = o \end{cases}$$

Thus G(d)=-oT

Combine (1) & (2) together, we get:

$$\mathrm{D} = \mathrm{G(o)} - \mathrm{G(d)} = (1-\mathrm{p})\{\mathrm{wW} - \mathrm{h(H+r)} - \mathrm{t(T+r)}\} - \mathrm{pH} + \mathrm{oT}$$

If D>0, it means that the player will have more income if he plays offense.

If D<0, it means that the player will have more income if he plays defense.

According to the formula, we can only decide whether we should attack or defend, however, we cannot choose which tile we discard and which combination we should make since the formula doesn’t give any hint on any specific tiles.

Hence, we decide to investigate on different attack situations, which result in different combinations of tiles.

Mathematical Calculations & Analysis

Assumptions:

1. All the tiles are randomly distributed to each player.
2. Tiles are drawn and discarded randomly by each player.
3. Ignoring the effect of riichi and dora on the scoring points.
4. Ignoring the effect of other players' decisions.
5. Only considering simple probability, not considering conditional probability.

Definitions for mathematical models:

111 refers to a Pong (3 identical tiles)

123 refers to a Chow (3 tiles in absolute numerical sequence and all in the same suit)

11 refers to a pair (2 identical tiles)

Situation 1: Big Three Dragons (containing 3 Haku, 3 Hatsu and 3 Chun)

Mathematical Models: 11 123 111 111 111

11 111 111 111 111

Since the player who has achieved 'Big Three Dragons' which must contain 3 Haku, 3 Hatsu and 3 Chun in his hands, 9 tiles are already determined.

The probability of a player getting these 9 tiles is $\frac{1}{4^9}$ (not considering other player's tiles)

There are 136 tiles in total, a player can win only if the tiles on all the player's hand reach

$14 + 13 \times 3 = 53$. Therefore, there are $C(53,136)$ kinds of combinations.

There are $\mathrm{C}\ (44{,}127)$ kinds of combinations for the remaining $53 - 9 = 44$ tiles

The probability of a player getting these 9 tiles is $\frac{\mathrm{C}(44{,}127)\times 4^3}{\mathrm{C}(53{,}136)} \times \frac{1}{4^9}$ (considering other player's tiles)

P1: Probability of (123)

There are 7 kinds of combinations $(1{,}2{,}3\ \ 2{,}3{,}4$ to $7{,}8{,}9)$, each number has 4 tiles with 3 different colors, we need 3 consecutive tiles.

Total number of tiles that can be selected are $127 - 16 - 3 = 108$

(16 wind tiles and 3 dragon tiles)

Hence, the probability of 3 tiles being consecutive in numbers (123) is $\frac{\mathbf{7\times 4^3\times 3}}{\mathbf{C(3{,}108)}}$

P2: Probability of (111)

For number tiles, there are 9 different numbers and 3 different colors, with 4 tiles of each kind. We need to select 3 tiles from the 4.

For wind tiles, there are 4 kinds (East, South, West, North), with 4 tiles each kind. We need to select 3 tiles from the 4.

Total number of tiles that can be selected are $127 - 3 = 124$ (3 dragon tiles have already been considered)

Hence, the probability of 3 tiles being the same (111) is $\frac{\mathbf{9\times C(3{,}4)\times 3+4\times C(3{,}4)}}{\mathbf{C(3{,}124)}}$

P3: Probability of (11)

For number tiles, there are 9 different numbers and 3 different colors, with 4 tiles each kind. We need to select 2 tiles from the 4.

Besides, the number tiles we select for (11) cannot be the same number as (111), so we need to cancel the repeated cases.

For wind tiles, there are 4 kinds (East, South, West, North), with 4 tiles each kind. We need to select 2 tiles from the 4.

Total number of tiles that can be selected are $127 - 3 = 124$ (3 dragon tiles have already been considered)

Hence, the probability of the remaining 2 tiles being the same (11) is

$$\frac{\mathbf{9 \times C(2,4) \times 3 - 4 + 4 \times C(2,4)}}{\mathbf{C(2,124)}}$$

The probability of getting the model '11 123 111 111 111' is $P_1 \times P_3$

The probability of getting the model '11 111 111 111 111' is $P_2 \times P_3$

Both of them occur at a probability of $\frac{1}{2}$

Hence, the probability of a certain player getting 'Big Three Dragons' patterns is

$$\frac{\mathbf{C(44,127) \times 4^3}}{\mathbf{C(53,136)}} \times \frac{\mathbf{1}}{\mathbf{4^9}} \times \mathbf{4} \times \left[\frac{\mathbf{7 \times 4^3 \times 3}}{\mathbf{C(3,108)}} \times \frac{\mathbf{9 \times 4 \times 3 + 4 \times 4}}{\mathbf{C(3,124)}} + \frac{\mathbf{7 \times 4^3 \times 3}}{\mathbf{C(3,108)}} \times \frac{\mathbf{9 \times 6 \times 3 - 4 + 4 \times 6}}{\mathbf{C(2,124)}}\right]$$

$\mathbf{= 2.062554694 \times 10^{-11}}$

Situation 2: Little Four Winds (Containing 3 groups of the wind tiles and a pair of the fourth kind)

Mathematical Models: 11 123 111 111 111

11 111 111 111 111

Since there are 4 kinds of wind tiles in total, the combinations of a player getting **3×3 identical wind tiles is** $\mathbf{C(3,4)} \times [\mathbf{C(3,4)}]^{\mathbf{3}}$

P1, P2 &P3 in situation 2 are exactly the same as in situation 1. This is because the mathematical models for situation 1&2 are the same, the only difference is the predetermined combinations of tiles "111 111 111"

Applying P1, P2&P3, we find **the probability of a certain player getting 'Little Four Winds' patterns is**

$$\frac{\mathbf{C(3,4)\times[C(3,4)]^3\times C(44,127)}}{\mathbf{C(53,136)}} \times \frac{\mathbf{1}}{\mathbf{4^9}} \times \mathbf{4} \times \left[\frac{\mathbf{7\times4^3\times3}}{\mathbf{C(3,108)}} \times \frac{\mathbf{9\times4\times3+4\times4}}{\mathbf{C(3,124)}} + \frac{\mathbf{7\times4^3\times3}}{\mathbf{C(3,108)}} \times \frac{\mathbf{9\times6\times3-4+4\times6}}{\mathbf{C(2,124)}}\right] = \mathbf{8.26218774\times10^{-11}}$$

Situation 3: Small Three dragons (containing 2×3 dragen tiles)

Mathematical Models: 11 111 111 123 123

11 111 111 111 123

11 111 111 111 111

When a player wins the game, he would have two sets of dragon tiles. Hence, there are $C(3,4) \times [C(3,4)]^3$ types of combination for his dragen tiles.

Besides these two sets of dragon tiles, there are still 130 tiles. Hence, the number of ways of combination is $C(45,130)$

P1: Probability of (123)

There are 7 kinds of combinations (1,2,3 2,3,4 to 7,8,9), each number has 4 tiles with 3 different colors, we need 3 consecutive tiles.

Total number of tiles that can be selected are $3 \times 9 \times 4 = 108$

Hence, the probability of 3 tiles being consecutive in numbers (123) is $\boldsymbol{\frac{7\times 4^3\times 3}{C(3,108)}}$

P2: Probability of (111)

For number tiles, there are 9 different numbers and 3 different colors, with 4 tiles of each kind. We need to select 3 tiles from the 4.

For wind tiles, there are 4 kinds (East, South, West, North), with 4 tiles each kind. We need to select 3 tiles from the 4.

Also, there would be four more dragon tiles which were not used before.

Total number of tiles that can be selected are $3 \times 4 \times 9 + 4 \times 4 + 4 = 128$

Hence, the probability of 3 tiles being the same (111) is $\boldsymbol{\frac{9\times 4\times 3+4\times 4+4}{C(3,128)}}$

P3: Probability of (11)

For number tiles, there are 9 different numbers and 3 different colors, with 4 tiles each kind. We need to select 2 tiles from the 4.

Besides, the number tiles we select for (11) cannot be the same number as (111), so we need to cancel the repeated cases.

For wind tiles, there are 4 kinds (East, South, West, North), with 4 tiles each kind. We need to select 2 tiles from the 4.

Also, there would be 4 more dragon tiles which can also be used as (11)

Total number of tiles that can be selected are 128

Hence, the probability of the remaining 2 tiles being the same (11) is $\frac{\mathbf{9 \times 6 \times 3 - 4 + 6 + 6 \times 4}}{\mathbf{C(2,128)}}$

$$\mathbf{P_1} = \frac{\mathbf{7 \times 4^3 \times 3}}{\mathbf{C(3,108)}}$$

$$\mathbf{P_2} = \frac{\mathbf{9 \times 4 \times 3 + 4 \times 4 + 4}}{\mathbf{C(3,128)}}$$

$$\mathbf{P_3} = \frac{\mathbf{9 \times 6 \times 3 - 4 + 6 + 6 \times 4}}{\mathbf{C(2,128)}}$$

The probability of getting the model '11 123 123 111 111' is $\mathbf{P_1 \times P_1 \times P_3}$

The probability of getting the model '11 123 111 111 111' is $\mathbf{P_1 \times P_2 \times P_3}$

The probability of getting the model '11 111 111 111 111' is $\mathbf{P_2 \times P_2 \times P_3}$

Hence, the probability of a certain player getting ‘Small Three Dragons' patterns is

$$\frac{C(2,3)\times[C(3,4)]^2\times C(47,130)}{C(53,136)}\times\frac{1}{4^6}\times 4\times[\frac{7\times4^3\times3}{C(3,108)}\times\frac{7\times4^3\times3}{C(3,108)}\times\frac{9\times6\times3-4+6+6\times4}{C(2,128)}+\frac{7\times4^3\times3}{C(3,108)}\times\frac{9\times4\times3+4\times4+4}{C(3,128)}\times\frac{9\times6\times3-4+6+6\times4}{C(2,128)}+\frac{9\times4\times3+4\times4+4}{C(3,128)}\times\frac{9\times4\times3+4\times4+4}{C(3,128)}\times\frac{9\times6\times3-4+6+6\times4}{C(2,128)}] = 1.45532457\times10^{-10}$$

General Situation

Mathematical Models: 11 123 123 123 123

11 111 123 123 123

11 111 111 123 123

11 111 111 111 123

11 111 111 111 111

First of all, there are a total of 136 tiles and 53 tiles that people hold when a player wins the game. Hence, there are C(53,136)kinds of situations that present when a player wins

P1: Probability of (123)

There are 7 kinds of combinations (1,2,3 2,3,4 to 7,8,9), each number has 4 tiles with 3 different colors, we need 3 consecutive tiles.

Total number of tiles that can be selected are $3 \times 4 \times 9 = 108$

Hence, the probability of 3 tiles being consecutive in numbers (123) is $\frac{\mathbf{7 \times 4^3 \times 3}}{\mathbf{C(3,108)}}$

P2: Probability of (111)

For number tiles, there are 9 different numbers and 3 different colors, with 4 tiles of each kind. We need to select 3 tiles from the 4.

For wind tiles, there are 4 kinds (East, South, West, North), with 4 tiles each kind. We need to select 3 tiles from the 4.

For dragon tiles, there are 3 kinds (Haku, Hastu and Chun), with 4 tiles of each kind. We also need to select 3 tiles from the 4.

Hence, the total number of tiles that can be selected are 136.

Hence, the probability of 3 tiles being the same (111) is $\frac{\mathbf{9 \times 4 \times 3 + 4 \times 4 + 3 \times 4}}{\mathbf{C(3,136)}}$

P3: Probability of (11)

For number tiles, there are 9 different numbers and 3 different colors, with 4 tiles each kind. We need to select 2 tiles from the 4.

Besides, the number tiles we select for (11) cannot be the same number as (111), so we need to cancel the repeated cases.

For wind tiles, there are 4 kinds (East, South, West, North), with 4 tiles each kind. We need to select 2 tiles from the 4.

For dragon tiles, there are 3 kinds (Haku, Hastu and Chun), with 4 tiles of each kind. We also need to select 2 tiles from the 4.

Hence the total number of tiles that can be selected are 136

Hence, the probability of the remaining 2 tiles being the same (11) is $\frac{\mathbf{9\times6\times3-4+6\times4+6\times3}}{\mathbf{C(2,136)}}$

$$\mathbf{P_1 = \frac{7 \times 4^3 \times 3}{C(3,108)} = \frac{112}{17013}}$$

$$\mathbf{P_2 = \frac{9 \times 4 \times 3 + 4 \times 4 + 3 \times 4}{C(3,136)} = \frac{1}{3015}}$$

$$\mathbf{P_3 = \frac{9 \times 6 \times 3 - 4 + 6 \times 4 + 6 \times 3}{C(2,136)} = \frac{10}{459}}$$

The probability of getting the model **‘11 123 123 123 123’** is $(P_2)^4 \times P_3$

=4.09200162×10^-11^

The probability of getting the model **‘11 111 123 123 123’** is $P_1 \times (P_2)^3 \times P_3$
$=2.06163301\times10^{-12}$

The probability of getting the model **‘11 111 111 123 123’** is $(P_1)^2 \times (P_2)^2 \times P_3$

$=1.03869233\times10^{-13}$

The probability of getting the model **'11 111 111 111 123'** is $(P_1)^3 \times P_2 \times P_3$

=5.23314159×10-15

The probability of getting the model **'11 111 111 111 111'** is $(P_1)^4 \times P_3$

=2.63656237×10-16

Hence, the probability of a certain player getting General patterns (a player wins the game in a general form) is

$$\begin{aligned}
4 \times \{ & [\frac{9\times4\times3+4\times4+3\times4}{C(3,136)}]^4 \times \frac{9\times6\times3-4+6\times4+6\times3}{C(2,136)} \\
& + [\frac{9\times4\times3+4\times4+3\times4}{C(3,136)}]^3 \times \frac{7\times4^3\times3}{C(3,108)} \\
& \times \frac{9\times6\times3-4+6\times4+6\times3}{C(2,136)} \\
& + [\frac{9\times4\times3+4\times4+3\times4}{C(3,136)}]^2 \times [\frac{7\times4^3\times3}{C(3,108)}]^2 \\
& \times \frac{9\times6\times3-4+6\times4+6\times3}{C(2,136)} \\
& + \frac{9\times4\times3+4\times4+3\times4}{C(3,136)} \times [\frac{7\times4^3\times3}{C(3,108)}]^3 \\
& \times \frac{9\times6\times3-4+6\times4+6\times3}{C(2,136)} \\
& + [\frac{7\times4^3\times3}{C(3,108)}]^4 \times \frac{9\times6\times3-4+6\times4+6\times3}{C(2,136)} \} = \mathbf{1.72364061\times10^{-10}}
\end{aligned}$$

Discovery

1.Situation 1&2 have more than 5 han and reach yakuman (the highest scoring points), whereas situation 3 only has 2 han

2.The han won by general situation is varying. Most of the cases, it can only get 1 or 2 han.

3.The probability of each situation:

$$1.72364061 \times 10^{-10} > 1.45532457 \times 10^{-10} > 8.26218774 \times 10^{-11} > 2.062554694 \times 10^{-11}$$

P(general situation) > P(situation 3) > P(situation 2) > P(situation 1)

Conclusions

1.Combinations that have higher scoring points are less likely to be formed.

2. If we only consider winning the game, the general situation is more preferable since its probability of winning is the highest. Within the general situation, the model '11 123 123 123 123' has the highest probability of winning the game.

3. However, if we add the overall scoring points into consideration, situation 2 (Little Four Winds) can give us the highest winning point and relatively high winning probability.

References

1.http://mahjong-europe.org/portal/images/docs/Riichi-rules-2016-EN.pdf

2.https://mahjong.guide/a-beginners-guide-to-riichi-mahjong/

3.http://arcturus.su/wiki/List_of_yaku

4.https://www.reddit.com/r/ffxiv/comments/adubo3/riichi_mahjong_basic_strategy/

5.http://beginners.biz/kihon/