Math Project Written Report

“The bigger the better”

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Our Objective

We wish to be able to find the largest possible square inscribed inside any distinct triangle.

We wish to derive a formula to compute the area of the largest possible square inscribed based on the dimensions of the triangle.

We also wish to be able to know, through some generalised rule, in what way would the square's area be maximised in certain triangles.

All these objectives would be closely linked with each other and we believe that finding one would intuitively lead to the other.

Methodology

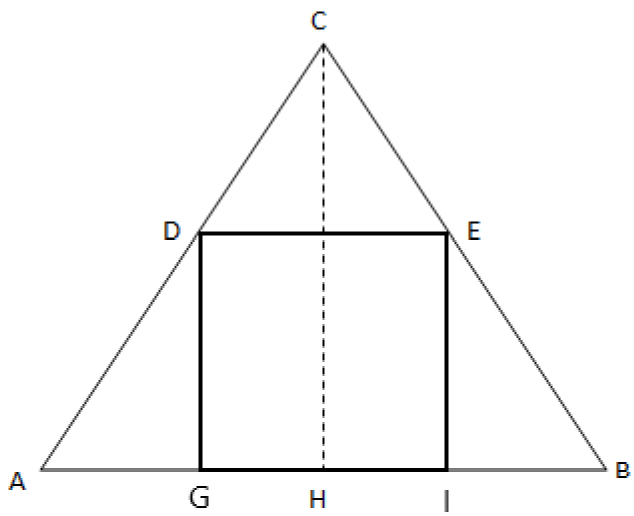
Firstly, we will self study on geometry and trigonometry and also search for and understand related studies. Then, we would derive a general formula and develop a general proof. After that, we would further our research based on interesting observations through exploration.

We would systematically work on deriving this formula by working with triangles in increasing levels of difficulty, starting with more regular triangles such as the equilateral, isosceles and right angled triangles, and moving to more irregular triangles such as scalene triangles, which would conclude our project's objective. We ranked the triangles in these degree of difficulty because of the variety of placements of squares. Naturally, the larger the variety of possible placements, the more tedious it would be to work on these triangles. Hence we work in ascending orders of difficulty-equilateral would only have one way of placing the square as all 3 sides of it are equal, yielding exact 3-way symmetry across. Isosceles yield two placements as it has a two way symmetry across its altitude. The right triangle is very unique as it does not (usually) offer a line of symmetry but its right angle coincides with the square's placement, hence there are also only two ways of placing. We start off with these 3 types in order to use some findings we have to assist us in working on some more tedious triangles.

Side note: We have found out that deriving the formula for most scalene acute triangles was of relative ease, but the conditions required were rather numerous. We aim to minimise the conditions needed for future formulas, however it serves as the scaffold for our research on regular triangles which in turn lead to progress on the scalene acute triangles with lesser conditions.

The flow in our agenda was slightly distorted though majority remains the same

Deriving the formula for equilateral triangles



$Let\ CD\ =\ x\ and\ DA\ =\ y$

$CD\ =\ DE\ =\ DG\ =\ x$

$AG\ =\ \sqrt{y^2 - x^2},\ for\ \angle DGA\ =\ 90°$

$AG\ =\ IB$

$2\sqrt{y^2 - x^2} + x\ =\ x + y$

$2\sqrt{y^2 - x^2} = y$

$4y^2 - 4x^2\ =\ y^2$

$3y^2 - 4x^2 = 0$

$3y^2 = 4x^2$

$y\sqrt{3} = 2x$

$y : x\ =\ 2 : \sqrt{3}$

$Let\ one\ side\ of\ the\ equilateral\ triangle\ be\ s$

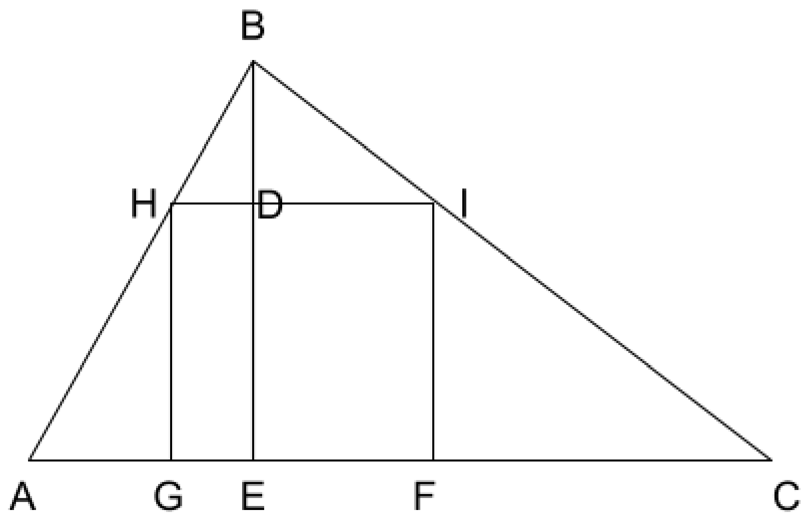
$Hence\ x\ = \frac{\sqrt{3}s}{2+\sqrt{3}}$

$$Largest\ square\ inscribed\ = x^2 = \frac{3s^2}{5+4\sqrt{3}}$$

Analysis of formula derived

Note that the formula derived over here only requires the length of one side of the triangle to be derived. This is because it is an equilateral triangle and all the angles are similar. As we progress on, these similar angles would not exist and thus it is less probable that we can progress on with deriving the formula for other triangles in this manner.

Deriving a more effective formula for irregular triangles



It would be more probable to derive a formula for irregular triangles without focusing only on the lengths on the side of the triangle, but focusing on using the base and height of the triangles to derive an algebraic equality for the area of the triangle, which in turn can be used to derive the area of the square.

$Let\ one\ side\ of\ square\ HIFH\ to\ be\ x$

$Let\ the\ altitude\ of\ triangle\ ABC\ (relative\ to\ AC),\ length\ BE\ be\ y$

$Let\ the\ base\ of\ the\ triangle\ ABC,\ ACbe\ z.$

$The\ area\ of\ the\ large\ triangle\ equates\ to\ the\ area\ of\ the\ square\ and\ 3\ small\ triangles$

$\frac{1}{2}zy = x^2 + \frac{1}{2}(x)(y - x) + \frac{1}{2}(z - x)(x)$

$\frac{1}{2}zy = x^2 + \frac{1}{2}(xy - x^2) + \frac{1}{2}(xz - x^2)$

$zy = 2x^2 + xy - x^2 + xz - x^2$

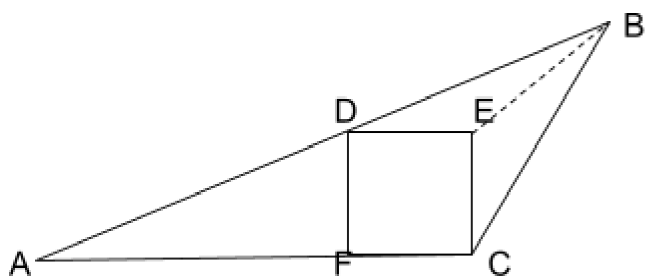
$zy = x(y + z)$

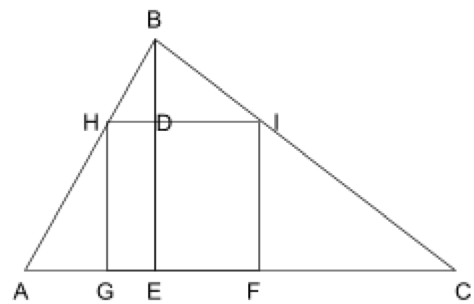
$$x = \frac{yz}{y+z}$$

$$x^2 = \frac{(yz)^2}{(y+z)^2}$$

Analysis of formula derived

This formula would be much more feasible for usage in most triangles. It only requires knowledge of the base and height of this triangle. However, some shortcomings of this formula would include things such as the inability to work for obtuse triangles:

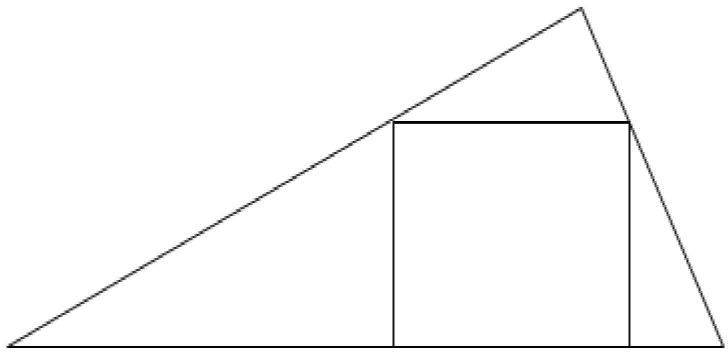




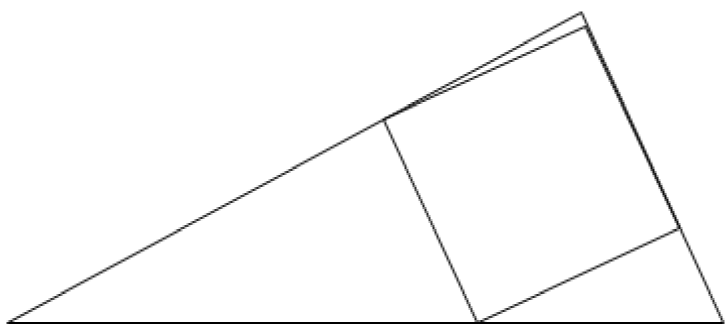
Here is a side by side comparison of the obtuse triangle with a square placed inside such that the side of the square lies on a side which is adjacent to an obtuse angle on the left, and a scalene acute triangle (the triangle which can be covered in this formula)

The simplest explanation here would be simply because this obtuse triangle would not have all 4 sides of the square touching the three sides of the triangle, rendering the current formula useless. More in depth analysis of obtuse triangles would be done in the Annex "Deriving the formula for all scalene obtuse triangles."

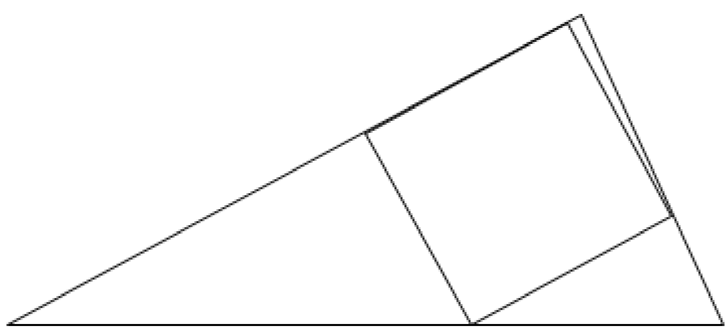
Moreover, we can see that there can be 3 ways in which a square can be placed in scalene acute triangles, each with the square on one side of the triangle:



Placement 1



Placement 2

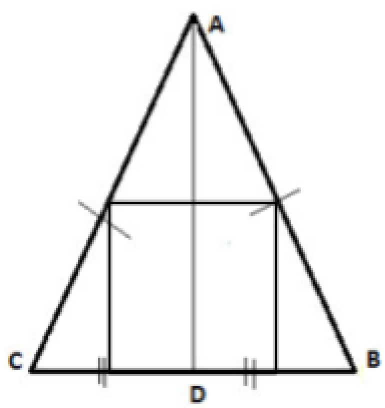


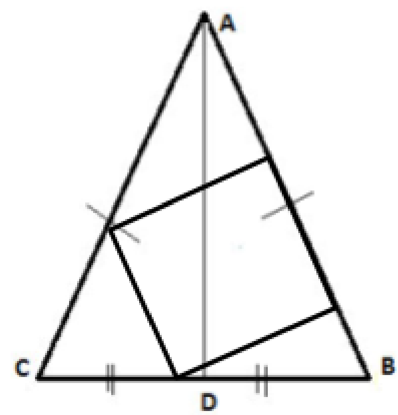
Placement 3

We note that these three placements would yield squares of differing sizes. The formula derived above can only be used for one placement to compute the area of the square, and cannot determine which placement would yield the square of largest area. Those are some of the things we would explore for this project.

Deriving the formula for right-angled and isosceles triangles

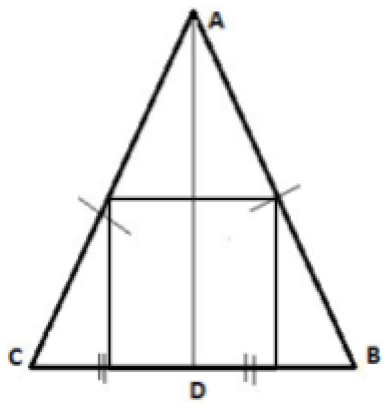
Isosceles Triangles





In this research on isosceles triangles and
the placement of squares inside of it, we would be only alternating one variable for this research and that is the altitude of the triangle relative to base CB, regardless of the placement of the square in the triangle itself. We know that as an isosceles triangle is symmetrical along this height, there can be two distinct placements of square inside, namely along the base CB (on the left) and along the slanted length AC/AB (on the right) (which are similar due to symmetry)

Type 1 placement: On side BC, altitude of triangleABC



Assume the base is 1. We will only be changing one variable, which is the altitude, h, of the triangle where h ∈ Z^{+}

H is also the angle bisector of ∠ABC due to the assumption that AB = BC. It will also intersect AC at 90°

Let h be length AD

Using pythagoras' theorem, $AB = BC = \sqrt{h^2 + (0.5)^2} = \sqrt{h^2 + 0.25}$

By using the previous results from deriving a more effective formula for irregular triangles,

$AB = \sqrt{h^2 + 0.25}$

$AC = 1$

Height = h

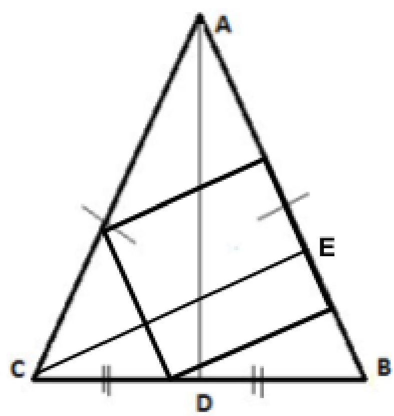
And using the formula,

Side of square = $\frac{h}{1+h}$

Naturally, the larger the side of the square, the larger the area of the square.

As h approaches infinity, side of square approaches 1*, area of square* *approaches* 1

Type 2 placement: On side AB, side



On the side AC or AB. the side which has the same length as another side.
Placing on AB yields a mirror image, hence we can just consider one case.

Let the side of this square be x.
We drop a perpendicular line from A to intersect BC at 90°, calling the point E. This would be the height in our formula, the base being CB.

Using pythagoras' theorem,
EA

$$= \sqrt{(\sqrt{h^2 + 0.25})^2 - x^2}$$

$$= \sqrt{h^2 + 0.25 - x^2}$$

Similarly, as ∠CEB = 90°,

$$CD = \sqrt{1^2 - x^2}$$

$$= \sqrt{1 - x^2}$$

We can form an algebraic equation:

$$\sqrt{(\sqrt{h^2 + 0.25})^2 - x^2} + \sqrt{1 - x^2} = \sqrt{h^2 + 0.25}$$

After computation that $x = \frac{2h}{\sqrt{4h^2+1}}$

We can substitute that into the formula derived.

$$Side\ of\ square = \frac{(\sqrt{h^2+0.25})(\frac{2h}{\sqrt{4h^2+1}})}{(\sqrt{h^2+0.25})+(\frac{2h}{\sqrt{4h^2+1}})}$$

$$= \frac{(\sqrt{h^2+0.25})(\frac{h}{\sqrt{h^2+0.25}})}{(\sqrt{h^2+0.25})+(\frac{h}{\sqrt{h^2+0.25}})}$$

$$= \frac{h}{\frac{h^2+h+0.25}{\sqrt{h^2+0.25}}}$$

$$= \frac{h}{\frac{(h+0.5)^2}{\sqrt{h^2+0.25}}}$$

We shall compare this value with the previous derived value when the square is placed on the base BC. It yields the side length of $\frac{h}{h+1}$

By comparing $\frac{h}{\frac{(h+0.5)^2}{\sqrt{h^2+0.25}}}$ and $\frac{h}{h+1}$, we can derive a range for h where one side is larger than another in order to compute the triangles to suffice for each condition.

As the numerator is identical, we can compare the denominator. When the denominator is larger, the value of the side of the square decreases, hence by comparing the denominator and reversing the results, we can see in which cases would either set-up of the squares be larger, or smaller conversely.

$\frac{(h+0.5)^2}{\sqrt{h^2+0.25}} against\ h + 1$

$\Leftrightarrow \frac{(h+0.5)^4}{h^2+0.25}\ against\ (h + 1)^2$

$\Leftrightarrow (h + 0.5)^4 against\ (h + 1)^2 (h^2 + 0.25)$

$\Leftrightarrow h^4 + 2h^3 + 1.5h^2 + 0.5h + 0.0625\ against$

$h^4 + 2h^3 + 1.25h^2 + 0.5h + 0.25$

$\Leftrightarrow 0.25h^2 against\ 0.1875$

$\Leftrightarrow h^2 against\ 0.75$

Note that we are here comparing the denominator of the fraction which is the side of the square.

Hence when $h^2 > 0.75\ or\ h > \sqrt{0.75,}$ the placement for the square on the base of the square is larger

When $h^2 < 0.75\ or\ h < \sqrt{0.75}$, the placement of the square on the slanted edge would yield a square of larger area.

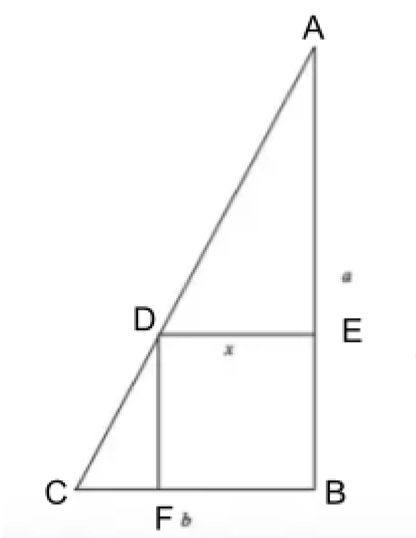
We note that interestingly, when equality holds, $h^2 = 0.75$,pythagoras states that the slant edge is $\sqrt{0.75 + (\frac{1}{2})(1 <-- base\ length)^2} = 1$, which is the same as the base. In this case, the triangle is equilateral; both placements are equally good.

Right-angled Triangles

Due to right angled triangles having a right angle, there is also only two placements as the 90 degree angle would coincide with the 90 degree angle on the square, leading to only two placements possible

The first possible placement

We can take a different approach for both types of placements in the context of the right angled triangles and that is to use the similar triangles



We can tell from this placement that

$\triangle ADE \sim \triangle ACB$ as

$\angle AED = \angle ABC, \angle DAE = \angle CAB$

$Similarly, \triangle CDF \sim \triangle CAB$

$Let\ DE = x,\ AB = y,\ BC = z$

$First\ pair\ of\ similar\ triangles\ yield: \frac{x}{z} = \frac{y-x}{y}$

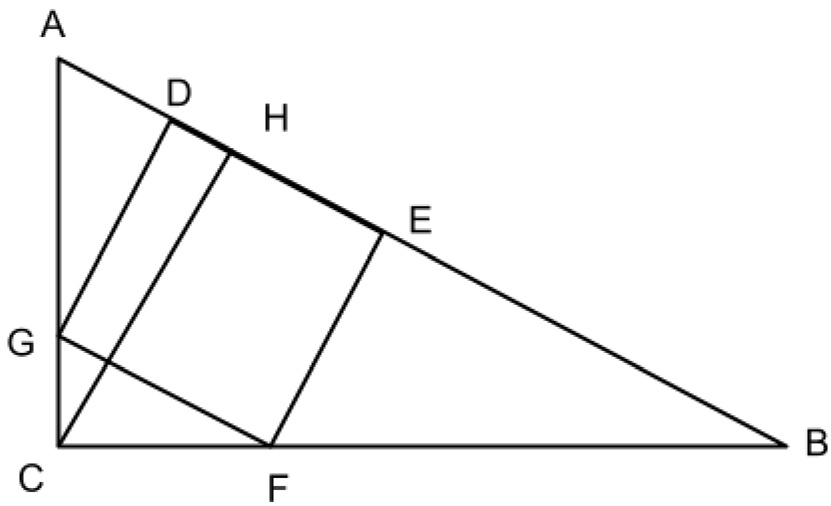
$xy = yz - xz$

$x = \frac{yz}{y+z}$

This formula is similar to the initial formula we had derived for most acute triangles at the start. However the deviation process is different, and the further use of this in our later research also varies. Basically, we are now focusing less on using the area to derive the formula but more on the property of similar triangles, which is present always as the square would create a parallel line for these triangles to be created.

Side note: Note how that would further increase the difficulty for obtuse triangles as the square may not necessarily create a parallel line (more on that in the obtuse triangles section)

The second possible placement is on the hypotenuse of the right angled triangle.



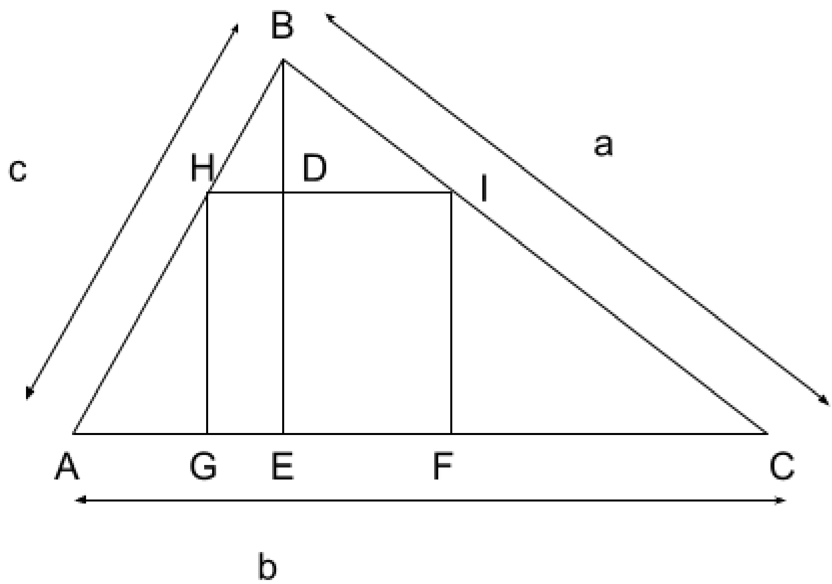
We find that this is quite similar to the previous set up and the context for the formula we have derived earlier, as the square forms two parallel lines. Note that this formation is not mutually exclusive to right-angled triangles, but it is simply easier to visualise with the earlier example.

$Letting\ CH\ be\ a,\ AB\ be\ b$

$We\ derive\ length\ = \frac{ab}{a+b}\ (from\ earlier\ formula)$

We observe that (in the previous example, and the current one) $ab\ =\ yz$as they are both twice the area of the triangle, which is $\frac{1}{2}bh$where those values are “bh”. Hence, the comparison between these values would be much easier. We would put all this knowledge to use in the derivation, and the observations of which placements are best, in the next section on scalene acute triangles.

Deriving the formula for all scalene acute triangles



$Let\ one\ side\ of\ square\ HIFG\ to\ be\ x$

$Let\ the\ altitude\ of\ triangle\ ABC\ (relative\ to\ side)\ be\ h$

$Let\ the\ base\ of\ the\ triangle\ ABC,\ ACbe\ b.$

$$\frac{x}{b} = \frac{h_b - x}{h_b}$$

$$h_b x = bh_b - bx$$

$$(h_b + b)x = h_b b$$

$$x = \frac{bh_b}{b + h_b}$$

$$s = \frac{1}{2}(a + b + c)$$

Area of triangle ABC = $\sqrt{s(s - a)(s - b)(s - c)}$

Comparing the 3 placements:

$$\frac{ah_a}{a+h_a}\ against\ \frac{bh_b}{b+h_b}\ against\ \frac{ch_c}{c+h_c}$$

$$a + h_a\ against\ b + h_b\ against\ c + h_c$$

$$a + \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}\ against\ b + \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{b}\ against\ c + \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{b}$$

The smallest among these 3 means that the placement is the largest.

As

$$ah_a,\ bh_b,\ ch_c\ are\ all\ the\ same\ (numerator)\ (twice\ of\ total\ area)$$

We can determine which placement is the best by simply comparing

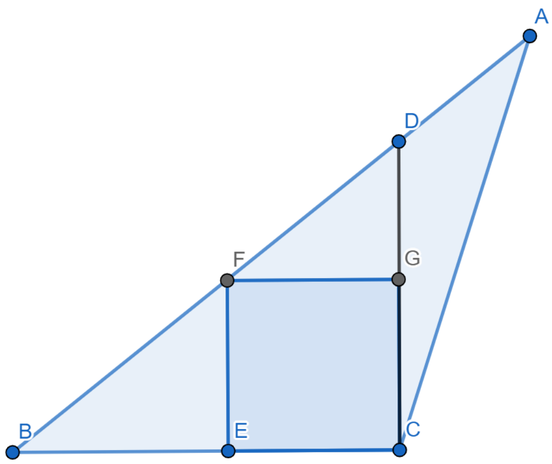
$$h_c + c\ \ against\ \ h_b + b\ \ against\ \ h_a + a$$

Whichever has the smallest value would imply the largest value for the side of the square.

Deriving the formula for all scalene obtuse triangles

We can divide these triangles into three cases, namely the squares lying on the three sides of these obtuse triangles.

Placement 1:



$Let\ the\ length\ of\ the\ side\ of\ the\ square\ FGCE\ be\ x.$

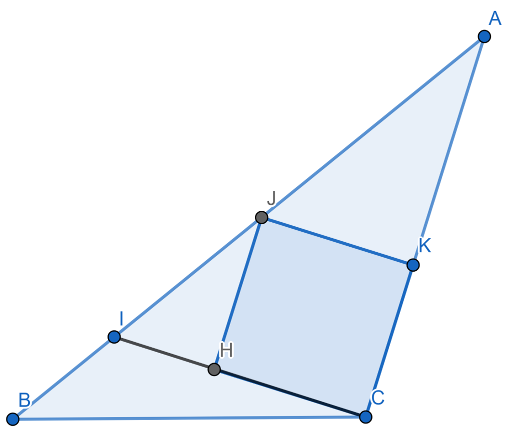
$Since\ the\ square\ is\ only\ in\ \triangle BCD,\ \triangle DAC\ is\ unnecessary\ in\ the\ calculation$

$Thus\ we\ can\ simply\ apply\ the\ formula\ for\ a\ right\ angled\ triangle\ derived\ above$

$$x = \frac{BC \times CD}{BC + CD},\ where\ CD = BC\ tan\angle B$$

$$\therefore x = \frac{BC(BC\ tan\ \angle B)}{BC + BC\ tan\ \angle B} = \frac{BC\ tan\ \angle B}{1 + tan\angle B}$$

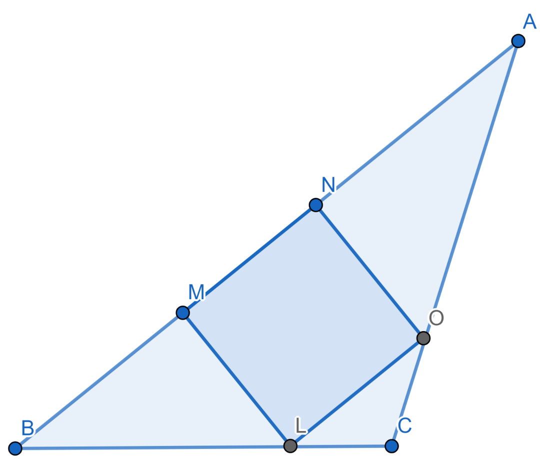
Placement 2:



Let the length of the side of the square CKJH be x.

Similar to placement 1, $x = \frac{AC \tan \angle A}{1+\tan\angle A}$

Placement 3:



Let the length of the side of the square ONML be x.

Let BA be c and h_c *be the altitude of side BA*

$Using\ the\ formula\ derived\ above\ for\ acute\ triangles,\ x = \frac{ch_c}{c+h_c}$

Comparison of the 3 placements:
Since the 3 formulae for the sides of the squares are very short and easy to use, plugging in the length and angles to the formulae and comparing their results will enable us to find the largest square in the triangle.

Obtuse isosceles triangles are just special cases, in which comparing placement 1 or 2 with placement 3 will yield the result as placement 1 is a mirror of placement 2.

Generalizations and conclusions

To conclude our findings, we have successfully formulas to calculate the sides of the squares and an easy way to compare this value to determine which placement would yield the largest square.

References

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