Spirals Into Finals! - Written Report

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1. Introduction

1.1 Project History

This entire project is based on the game “Spirals of Gold”, which is originated from the book “Blood Sword Volume 1”. It is not a common game among people, yet it is very interesting to play and discover.

1.2 Project Aims

By solving our project, we can:

- ★ using research, find out more about the scenarios in Spirals of Gold,
- ★ determine strategies in the game Spirals of Gold, and
- ★ make use of skills equipped from this project to solve other Combinatorics problems.

1.3 Gameplay

1.3.1 Basic Description

This game is played in rounds. In each round, both players act simultaneously. The players first choose a number, independently, that must be a positive integer strictly less than the number of heads the player currently has. The two players then compare their numbers, and thereafter, perform moves on the coins based on the comparison.

1.3.2 Required Items

All we need to play Spirals of Gold are 14 coins, with 7 coins dealt to each of the 2 players. (Refer to Figure 1)

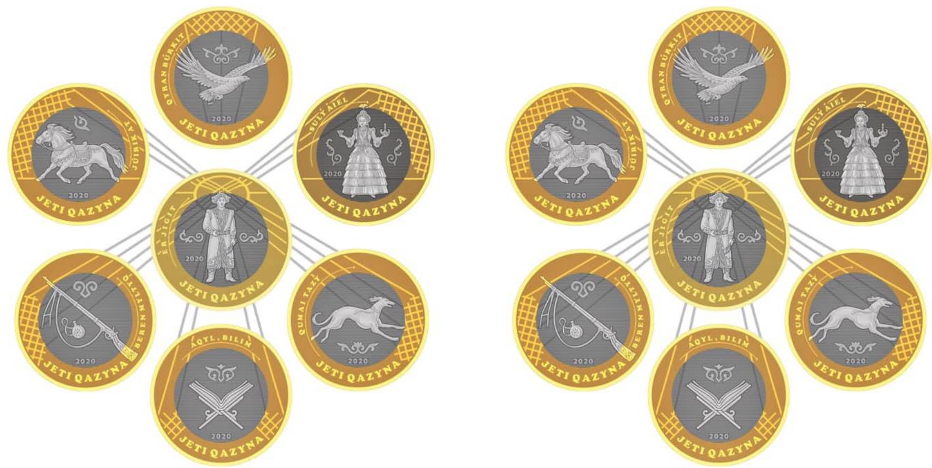

Figure 1 - 14 identical coins, split equally among both players

1.3.3 Playing The Game

Each player should have 7 coins at the start, and all of them should have heads facing up. (Refer to Figure 2)

Player 1

Player 2

Figure 2 - Starting Scenario

At the start of every round, if a player has a coin with tails facing up, he gets to flip one such coin from tails to heads. (Refer to Figure 3)

Figure 3 - Flipping 1 coin from tail to heads at the start of a round

Next, players each pick a number, which is strictly smaller than the number of heads that player has currently.

After this, the players perform moves in this manner.

➔ Both players do nothing if the numbers chosen are equal.
➔ The player choosing the larger number will flip that number of heads into tails. On the other hand, the player choosing the smaller number will discard a number of heads equal to the difference between the two numbers.

This continues until a player wins.

1.3.4 Success

A player only wins when the opponent has at most 1 head left, hence the opponent cannot perform any moves.

2. Research Questions Overview

The following are our 3 research questions:

- ★ RQ1 - Let $2 < n < 7$ be an integer. What is the strategy of winning the game if each player is given n coins?
- ★ RQ2 - Find the strategy of winning the game when $n = 7$, just like in the original game.
- ★ RQ3 - Find the probability of the strategy in RQ2 working.

We will present our solutions for the 3 research questions in this report.

3. RQ1 - Let $2 < n < 7$ be an integer. What is the strategy of winning the game if each player is given n coins?

3.1 Necessary Information

For this research question, n ranges from 3 to 6.

In addition, the number of heads a player has:

- ➔ is at least 2, else the player loses
- ➔ is at most n

Therefore the number of possible number of heads is $n - 1$.

Hence, the number of cases in this research question is calculated using

$$(2 + 3 + 4 + 5)^2$$

which has a result of 196.

3.2 Solving Process

Since there is no fixed strategy to solve the problem, we have decided to execute the solving of the 196 cases, one by one. We have set different scenarios and determined the best strategy for each case, throughout the few months of solving this problem.

Next, we have consolidated our findings into a table. Thereafter, we have come up with certain inferences and conclusions, based on the general strategies, and patterns for success and failure. Below is a preview of our table. (Refer to Figure 4)

Total coins/# of heads		YOU	2		3			4				5					6				
			2		2	3		2	3	4		2	3	4	5		2	3	4	5	6
Opponent	STRATEGY (Which number to pick)																				
2	2		1		1	2		1	2	3		1	2	3	4		1	2	3	4	5
3	2		lose		1	2		1	2	3		1	2	3	4		1	2	3	4	5
	3		lose		lose	2		1	2	3		1	2	3	4		1	2	3	4	5
4	2		lose		lose	tie		1	2	3		1	2	3	4		1	2	3	4	5
	3		lose		lose	lose		lose	2	3		1	2	3	4		1	2	3	4	5
	4		lose		lose	lose		lose	2	3		1	2	3	4		lose	2	3	4	5
5	2		lose		lose	lose		lose	2	3		1	2	3	4		1	2	3	4	5
	3		lose		lose	lose		lose	lose	3		lose	2	3	4		1/tie	2	3	4	5
	4		lose		lose	lose		lose	lose	lose		lose	lose	3	2		lose	2	3	2	5
	5		lose		lose	lose		lose	lose	lose		lose	lose	3	3		lose	lose	3	2	2/3
6	2		lose		lose	lose		lose	lose	lose		lose	lose	3	3/4		1	2	3	4	5
	3		lose		lose	lose		lose	lose	lose		lose	lose	lose	4		lose	2/tie	1	4	5
	4		lose		lose	lose		lose	lose	lose		lose	lose	lose	2		lose	tie	3/tie	2	5
	5		lose		lose	lose		lose	lose	lose		lose	lose	3	lose		lose	lose	3	4/tie	3
	6		lose		lose	lose		lose	lose	lose		lose	lose	lose	4		lose	lose	lose	4	3/4

Figure 4 - The final table of results we obtained

3.3 Strategy Analysis

The nature of the game allows for no fixed strategy, and it does not even guarantee that there is a strategy for every case. Hence, we will analyse a few cases that have a strategy and have no strategy.

3.3.1 Cases with Strategy

There are some cases with a strategy. An example of this would be in the figure below. (Refer to Figure 5)

Figure 5 - An example of a case with a winning strategy

In this case, there is a strategy because you can choose a 3. The opponent can only choose at most 2. If the opponent chooses a 2, you will get this case, which will guarantee a win provided that you use the optimal strategy.

3.3.2 Cases with no strategy

Of course, there are certain cases with no strategy at all. They either depend on your opponent, or guarantee a loss. The following shows a case in which your opponent has a strategy, and you will definitely lose. (Refer to Figure 6)

Figure 6 - An example of a case which has no winning strategy, and you definitely lose.

In this case, as you only have 2 heads, the only choice you have is 1. The opponent can play a 3, wiping all your heads. This makes you have only 1 head on the next turn, causing a loss.

3.4 Conclusions

Based on our findings, we have consolidated the results in a table. This was shown in our preview. (Refer to Figure 4, on Page 6)

Hence, among the 196 cases, 129 cases have a strategy of success.

In addition, interesting results were also obtained, as follows.

- ➢ Strategies usually follow the form $n - 1$, where n is the number of cards dealt to each player at first.
- ➢ The more heads one has, the higher the chance of winning.
- ➢ Generally, if both players have the same number of heads, the one with more tails is likely to win.
- ➢ A loss in total number of coins is not always a bad sign. This is because the opponent has less total playable heads, which could be an opportunity for you to win!

4. RQ2 - Find the strategy of winning the game when $n = 7$, just like in the original game.

4.1 Necessary Information

In this research question, the number of heads has increased to 7. Thus, the number of heads each player can play is from 2 to 6. Hence, there is an addition of

$$(2 + 3 + 4 + 5 + 6)^2 - 196 = 204$$

cases that we need to work on.

4.2 Solving Process

Our solving process is similar to that of RQ1. We will also use a table to consolidate all our findings, which are obtained from doing systematic listing of the cases.

4.3 Strategy Analysis

4.3.1 Cases with Usual Strategy

In this case, we define cases with a “usual strategy” as those cases which have a strategy, and its strategy is to play $n - 1$ when they have n heads at the moment.

An example of a case with the “usual strategy” is shown below.
(Refer to Figure 7)

Figure 7 - An example of a case, which uses the “usual strategy”.

In the case above, the strategy is to choose a 5. Since your opponent can only play 1 or 2, it forces your opponent to discard 3 or 4 coins which makes them have only 1 head for the rest of the game. This leads to a loss for your opponent.

4.3.2 Cases with Special Strategies

Now, we define cases with “special strategies” as those cases which do have a strategy, yet they do not follow the “usual strategy” in the previous section. The following shows an example. (Refer to Figure 8)

Figure 8 - An example of a case that requires a “special strategy”.

If we play according to our “usual strategy”, we will play a 4. But, our opponent can play a 3. After performing the moves, and flipping back one tail, the following shows the result. (Refer to Figure 9)

Figure 9a - The result of using the “usual strategy”.

With only 2 heads, you can only play 1 head from then on. This will result in the opponent gaining the upper hand, and winning the game by playing a 4.

Hence, we can conclude that not all the cases should be settled using the "usual strategy", though it is the best solution for most cases.

A wise move for this case (Refer to Figure 8) would be playing a 2 instead. For the sake of providing an example, we will show the result if the opponent still plays a 3. After performing the moves, and flipping back one tail, the following is the result. (Refer to Figure 9b)

Figure 9b - The result of using a "special strategy".

As concluded in RQ1, you are likely to win since you have more heads. This shows the power of using a "special strategy" at times instead of following the "usual strategy" which can be defeated sometimes.

4.4 Research Question Strategy

Coming back to the Research Question, it is apparently a case which requires a "special strategy" and the "usual strategy" does not work. After manual calculations, the strategy derived is playing a 5.

4.5 Conclusions

The direct answer to the question would be choosing a 5, and thereafter playing each round using the strategies that we have found, as shown in the table below. (Refer to Figure 10)

Total coins/# of heads		YOU	2	3		4			5				6					7					
			2	2	3	2	3	4	2	3	4	5	2	3	4	5	6	2	3	4	5	6	7
	STRATEGY (Which number to pick)																						
2	2		1	1	2	1	2	3	1	2	3	4	1	2	3	4	5	1	2	3	4	5	6
3	2		lose	1	2	1	2	3	1	2	3	4	1	2	3	4	5	1	2	3	4	5	6
	3		lose	lose	2	1	2	3	1	2	3	4	1	2	3	4	5	1	2	3	4	5	6
4	2		lose	lose	tie	1	2	3	1	2	3	4	1	2	3	4	5	1	2	3	4	5	6
	3		lose	lose	lose	lose	2	3	1	2	3	4	1	2	3	4	5	1	2	3	4	5	6
	4		lose	lose	lose	lose	2	3	1	2	3	4	lose	2	3	4	5	lose	2	3	4	5	6
5	2		lose	lose	lose	lose	2	3	1	2	3	4	1	2	3	4	5	1	2	3	4	5	6
	3		lose	lose	lose	lose	lose	3	lose	2	3	4	tie	2	3	4	5	1	2	3	4	5	6
	4		lose	lose	lose	lose	lose	lose	lose	lose	3	2	lose	2	3	2	5	lose	2	3	3	5	6
	5		lose	lose	lose	lose	lose	lose	lose	lose	3	3	lose	lose	3	2	2/3	lose	2/tie	3	4	4	6
6	2		lose	lose	lose	lose	lose	3	lose	lose	3	3/4	1	2	3	4	5	1	2	3	4	5	6
	3		lose	lose	lose	lose	lose	lose	lose	lose	lose	4	lose	2/tie	1	4	5	1/tie	2	3	4	5	6
	4		lose	lose	lose	lose	lose	lose	lose	lose	lose	2	lose	tie	3/tie	2	5	lose	2	3	4	5	6
	5		lose	lose	lose	lose	lose	lose	lose	lose	3	lose	lose	lose	lose	4/tie	3	lose	2	3	2	5	6
	6		lose	lose	lose	lose	lose	lose	lose	lose	lose	4	lose	lose	lose	4	3/4	lose	lose	3/tie	4	5	4
7	2		lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	3	4	5	1	2	3	4	5	6
	3		lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	5	lose	2/tie	3/tie	4	5	6
	4		lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	2/tie	2	2	5	6
	5		lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	3	3	3	3
	6		lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	4	3	3
	7		lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	lose	3	4	5	5

Figure 10 - The strategies for every case, from 2 to 7.

In addition, we have obtained more interesting findings:

- ➢ Strategies may not always be the “usual strategy”, but may require a “special strategy” that needs manual trial and error.
- ➢ Having more tails, while having the same number of heads, guarantees you a win.
- ➢ As seen in special cases, discarding coins is not a bad sign. Neither is choosing a large number of heads a bad sign too.

5. RQ3 - Find the probability of the strategy in RQ2 working, assuming the opponent picks a random legal number in each move.

5.1 Solving Process

Based on the strategy discussed in RQ2, we found out the cases that may arise if the opponent picks a number after we follow a certain strategy from a certain situation. For example, in the starting configuration, where each player has 7 heads out of 7 total coins, the strategy states that we should pick 5. Thus, we considered the different cases where the opponent picks 1, 2, 3, 4, 5 and 6 respectively, each with a 1/6 chance of happening. We continued to branch out at each situation, until we came to a definite conclusion that we will win or lose no matter what the opponent picks. We then sum up the probabilities of each case.

5.2 Findings

		Win	Draw	Loss
1,2,3,4,6 raw		0.749349	0.0423177	0.0416667
1,2,3,4,6 out of 1		0.8992188	0.0507812	0.05
5 tie chance			0.0046296	
5 non-tie chance			0.162037	
5 non-tie prob		0.1457067	0.0082284	0.0081019
	TOTAL	0.8950557	0.0551758	0.0497685
	in %	89.505573	5.517575	4.9768519

Figure 5: Probability of the strategy in RQ2 succeeding against an opponent playing randomly.

The second row, labeled “1, 2, 3, 4, 6 raw”, consists of the combined raw probabilities of winning, drawing and losing when the bot picks a 1, 2, 3, 4 or 6. Thus, the total of the numbers in that column is only ⅚.

The third row, labeled “1, 2, 3, 4, 6 out of 1”, consists of the data in the previous row, but processed such that their pairwise ratios remain the same, and the total of the three numbers is 1.

The fifth row, labeled “5 tie chance”, is the probability that the game ties at the beginning, which is equal to the probability that the opponent picks 3 fives in a row.

Note here that if the opponent picks any other number than 5 in the first three rounds, it does not matter how many 5s the opponent picked before this, because they do not alter the game situation.

The sixth row, labeled “5 non-tie chance”, is the chance of NOT drawing trivially, while the opponent still picked at least one 5. This is equal to the number of the previous row subtracted from ⅙, as this is the chance of the opponent picking a 5 for the first turn.

The seventh row, labeled “5 non-tie prob”, is the chance of getting a win, loss, and draw, where the opponent picked at least one 5. Note that the “draw” here does not include the trivial game of the opponent just picking three 5s in a row. It includes other situations of draw. As the game is unchanged no matter the number of 5s picked in the beginning, the numbers are simply equal to the probabilities in the third row, which sum up to 1, multiplied by the number in the previous row, the chance of this non-tie event happening in the first place.

Then, the total is shown. Our strategy has a win rate of 89.5% against a completely random opponent.

5.3 Bugs

This random opponent might just throw out his advantage simply because he picked a 1 on a particular round, whereas a normal opponent playing by judgement will never pick a 1. This bug is hard to prevent and we are, unfortunately, unable to simulate an opponent playing by judgement.
If the opponent is playing greedily, i.e. he always picks the largest number possible, the game will go in a fixed manner, and it can be calculated that the player with the strategy will win.

6. Overall Conclusion

RQ1 - The strategies are shown in a table. (Refer to Figure 4, on Page 6)
In particular,

- ➢ the strategy for $n = 3$ is to play a **2**.
- ➢ the strategy for $n = 4$ is to play a **3**.
- ➢ the strategy for $n = 5$ is to play a **3**.
- ➢ the strategy for $n = 6$ is to play a **3 or 4**.

RQ2 - The strategy is to play a **5**.

RQ3 - Our strategy in RQ2 has a win rate of **89.5%** against an opponent playing randomly.

7. Appendices

Here is the C++ code we used to simulate our game (instead of using real coins):

```
#include<iostream>
using namespace std;
int main()
{
    int Player_Head = 7, Opponent_Head = 7, Player_Total = 7, Opponent_Total = 7,
Player_Number, Opponent_Number;
    int Turn = 0;

    while (Player_Head > 1 && Opponent_Head > 1) {
        Turn++; // New Turn

        // Input numbers
        cout << "Turn " << Turn << "\n" << "Player number: ";
        cin >> Player_Number;
        cout << "Opponent number: ";
        cin >> Opponent_Number;

        // Invalid numbers
        if (Player_Number >= Player_Head || Opponent_Number >= Opponent_Head) {
               cout << "Error: Invalid number\n";
        }


        else {

               // Player larger
               if (Player_Number > Opponent_Number) {
                      Player_Head -= Player_Number;
                      Opponent_Head -= (Player_Number - Opponent_Number);
                      Opponent_Total -= (Player_Number - Opponent_Number);
               }
               else if (Player_Number == Opponent_Number); // Do nothing if same

               // Opponent larger
               else {
                      Opponent_Head -= Opponent_Number;
                      Player_Head -= (Opponent_Number - Player_Number);
                      Player_Total -= (Opponent_Number - Player_Number);
               }
        }

        // Add 1 head after each round
        Player_Head++;
        Opponent_Head++;

        // Checks if heads larger than total
        if (Player_Head > Player_Total) Player_Head = Player_Total;
        if (Opponent_Head > Opponent_Total) Opponent_Head = Opponent_Total;
```

```
        cout << "Outcome: Player: " << Player_Head << "/" << Player_Total << "; Opponent: " <<
Opponent_Head << "/" << Opponent_Total << "\n\n";
    }
    if (Player_Head < 2) cout << "Loss";
    if (Opponent_Head < 2) cout << "Win";
}
```

8. Possible extensions to the project

We can consider a few extensions.

- We can generalize the number of coins to n coins, for any integer n.
- We can consider the possibility of adding a third player inside the game, and in each round, only the players who picked the highest and lowest number flip over/discard their coins.