# Sharing isn't caring

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## Abstract

This projects aims to investigate how the addition of different conditions, namely a win condition and a punishment, would affect the Nash equilibrium of the public goods game (PGG), and how real players would react to this conditions. This project used the expected pay-off function to find the Nash equilibrium and prove it with mathematical logic and statistics generated by a computer algorithm. The introduction of a win condition does not affect the Nash equilibrium (The Nash equilibrium is to contribute the least amount possible), while the introduction of the punishment changed the Nash equilibrium (The Nash equilibrium is to contribute all the coins possible). Players responded differently with the addition of these conditions; the win condition resulted in players playing more selfishly while the penalty has encouraged players to contribute more (more than those players in the win condition variation).


# 1 Introduction

## 1.1 Brief Introduction

A public good is a good or service that is non-excludable (where it is difficult to ban people from using it and use is free) and non-rivalrous (where one person's usage of the good does not affect others usage). This encourages a free-loader behaviour among users of this public good: where they contribute the least to the public good, thus benefiting most from using it. It creates a social dilemma - a situation where individuals profit the most from selfishness but benefits the least if everyone chooses to be selfish, as there would be no contribution towards the creation of a public good.

The public goods game can be seen as a model of a common everyday example, such as climate change. Climate change is most likely the greatest social dilemma in history: each country and individual faces private costs to reduce greenhouse gas emissions, while the benefits of such mitigation efforts are shared by all countries and individuals, regardless of their own contributions. Hence, what is rational on individual and country levels is not globally optimal (Hasson, Löfgren, & Visser, 2010).

In the game, subjects are given a set amount of starting tokens. In the base game, there are a set number of rounds. In each round, players are anonymously given the choice of how to allocate their tokens; to either keep them or to put them in the shared bank. After all players have made their decisions, the total number of tokens in the shared bank is multiplied by a fixed multiplier, distributed equally among all the players to be added instantly into their personal tokens, and the next round commences. This process is repeated until the game concludes. However, the setting of this game is too unrealistic. In this project, more chaos is introduced into the game. This project aims to find the optimal choice to make, a.k.a. the Nash equilibrium (regardless of what other players choose) in the public goods game, after the addition of thresholds, penalties, and other factors

Lastly, this projects will compare the optimum strategy provided by the Nash equilibrium with real human behaviour in the different variation of the game.

## 1.2 Rationale

We were introduced to a variation of the game, known as the Greater Good Game, in the Japanese animated television programme, Kakegurui XX, which is adapted from the Kakegurui Japanese comic. In the series (Episode 7 to 8 or Chapter 46 to 49), 5 players were given 5 coins at the start of the round and the bank multiplier is at 2 times. The players need at least 40 coins at the end of the game to be qualified to win. However, the characters in the game cheated. We were interested in the general concept of the game and embarked on this project to explore more in this area and the limits provided in the series as they make the game more complicated.

## 1.3 Aim

This project aims to find how the introduction of new variables in the public goods game affects its Nash Equilibrium, and how real players play compared to this equilibrium.

## 1.4 Objectives

### 1.4.1 Part 1 (Theory)

1. To find the Nash equilibrium of the public goods game for any players, starting coins and rounds.

2. To find the Nash equilibrium of (1) with the addition of a win condition/threshold.

3. To find the Nash equilibrium of (2) with the addition punishing free loaders.

### 1.4.2 Part 2 (Empirical)

4. To compare the Nash equilibrium strategy of (6) with real human choices.

## 1.5 Research Problems

### 1.5.1 Part 1 (Theory)

1. What is the optimal allocation of tokens a player in the standard public goods game?

2. How does adding a win condition/threshold change a player's strategy?

3. How does punishing free loaders change a player's strategy?

### 1.5.2 Part 2 (Empirical)

4. Will real humans always follow the Nash equilibrium optimal strategy?

## 1.6 Fields of mathematics

1. Mathematics logic

2. Game theory in economics

3. Statistics/Computational economics

# 2 Literature review

## 2.1 Public goods

Public goods are defined as goods that are joint, equal, non-rivalrous consumption and non-excludability. Joint and non-rivalrous consumption are defined as one individual's consumption of a good (or service) does not prevent another individual from consuming the same good, and non-excludability is defined as any good such that, if any person $X$; in a group $X_1$, ... $X_i$, ... $X_n$ consumes it, it cannot feasibly be withheld from the others in that group (Cowen, 1985).

A threshold public good is provided if and only if total contributions towards its provision are sufficiently high. The classic example would be a capital project such as a new community school (Andreoni, 1998). The notion of threshold public good is, however, far more general than this classic example. Consider, for example, a charity that requires sufficient funds to cover large fixed costs. Or, consider a political party deciding whether to adopt a policy which is socially efficient but, for some reason, unpopular with voters; the policy will be enacted if and only if enough party members are willing to back the policy (Goeree and Holt, 2005).

In the public goods game, players autonomously and privately choose how many of their own coins to put into a public pot. The tokens in this pot are multiplied by a factor (greater than one and less than the number of players, N) and this "public good" payoff is evenly divided among players. Each subject also keeps the tokens they do not contribute (Wikimedia Foundation, 2020).

As such, the public goods game is a complete information, where each player is fully aware of the rules and utility function (preferences over a set of goods and services) of each of the players and imperfect information, where the player is not informed of every other player's choices, under the assumption that every player is rational and will not play strictly dominated strategies (Ferber, Luce, & Raiffa, 1959).

## 2.2 Game theory

Payoff refers to the payout a player receives from arriving at a particular outcome (McNulty, 2020).

An $N$-tuple of strategies $\langle \mathbf{p}_1, ..., \mathbf{p}_N \rangle$ for players 1, ..., $N$ respectively is called a Nash equilibrium if and only if for all $n$ and for all $\mathbf{p}$ ($\mathbf{p}$ a strategy for player $n$).

$$C^n(\mathbf{p}_1, ..., \mathbf{p}_{n-1}, \mathbf{p}, \mathbf{p}_{n+1}, ..., \mathbf{p}_N) \leq C^n(\mathbf{p}_1, ..., \mathbf{p}_N) \quad (1)$$

i.e., given that all other players $i \neq n$ use strategies textbfp$_1$, players $n$ best response is $\mathbf{p}_n$ (Leshem & Zehavi, 2008).

The Nash equilibrium is only motivated under the assumption that rational players do not play strictly dominated strategies. In the normal-form representation of $n$-player game specifies the players' strategy spaces: $S_1, ..., S_n$ and their payoff functions $u_i, ..., u_n$, the game is denoted by $G = \{S_1, ..., S_n; u_1, ..., u_n\}$. Let $s_i'$ and $s_i''$ be feasible strategies for players $i$, (i.e., $s_i'$ and $s_i''$ are members of $S_i$). Strategy $s_i'$ is strictly dominated by strategy $s_i''$ if for each feasible combination of the other players' strategies, $i$'s payoff from playing $s_i'$ is strictly less than i's payoff from playing $s_i''$ (Gibbons, 2013). The Nash equilibrium only exist when there is a finite number of pure strategies.

In situations where players would like to out guess others, the above definition of the Nash equilibrium, where one player's pure strategy strictly dominates another players' pure strategy, does not exist because the solution to such a game necessarily involves uncertainty about what other players will do. As such, a mixed strategy is used. In the normal-form game $G = \{S_1, ..., S_n; u_1, ..., u_n\}$, suppose $S_i = \{s_{i1}, ..., s_{iK}\}$. Then a mixed strategy for player $i$ is a probability distribution $p_i = (p_{i1}, ..., p_{iK})$, where $0 \leq p_{ik} \leq 1$ for $k = 1, ..., K$ and $p_{i1} + ... + p_{iK} = 1$. However, the Nash equilibrium can be found by extending the definition to require each players' mixed strategy to be the best response to another players' mixed strategy. I.e., $\sigma^*$ is a Nash equilibrium if $\sigma_i^* \in BR_i(\sigma_{-i}^*)$ for all players $i$, $BR$ meaning the set of best responses for player $i$ to $\sigma_{-1}$ (George, n.d.).

However, in most games, rationalizability does little to constrain players' behavior, and people may not follow the Nash equilibrium optimal strategy even if there is only one (Mailath, 1998).

# 3 Methodology

## 3.1 Definition of nomenclature

$P$ = number of players

$R$ = number of rounds

$C$ = number of starting coins for every round

$B$ = shared bank multiplier

$X$ = number of coins placed in personal bank

$Y$ = number of coins placed in shared bank

$M$ = minimum number of coins to win the game

## 3.2 Rules

1. $P$, $R$, $C$, $B$ and $M$, are decided before the game begins.

2. The order in which each player makes his/her choices is determined: $P_1, ..., P_i, ..., P_n$

3. Each player starts off every round with C coins.

4. Every round, according to the order predetermined, $P_i$ makes an secret and autonomous choice in how he/she wants to use his/her coins.

5. The coins must either go into $P_i$'s own personal bank or the shared bank only. No coins can be saved for future rounds.

6. After each round, the total number of coins are tallied and doubled, which is then equally split among the players, to be added immediately to the player's personal bank in the same round.

7. Process 3 to 6 repeats until all the rounds is over.

## 3.3 Methods

Below are the methods this project has used to investigate the research problems.

1. To find the Nash equilibrium defined in section 1.4 and prove it mathematically.
2. To use mathematics logic to explain the proof and to support it using a computer generated model of the game.
3. Conduct an experiment on real people and compare the results to the Nash equilibrium optimal solution.

Each condition is under the assumption defined in section 4.1.

# 4 Results and discussions

## 4.1 Assumptions

As players can make any choices they want, some assumptions have to be made.

1. Every player is a rational player trying to win every other players.
2. Every player is a rational player trying to maximise their utility.

This ensures that all players play the game properly and do not collude or cheat which will affect the result of the game.

## 4.2 Maximum number of coins

In this subsection, a formula to calculate the number of coins a player gets from the bank is investigated. This will facilitate the algorithm in the computing program to generate the possibilities. The number of coins each player receives from the bank is the sum of each players contribution, multiplied by the bank multiplier, divided by the number of players:

$$\frac{B\sum_{i=1}^{P} Y_i}{P} \tag{2}$$

Hence, to calculate the number of coins in the personal bank for player $P_i$, it can be expressed as:

$$\frac{B\sum_{i=1}^{P} Y_i}{P} + X_i \tag{3}$$

The number of coins in the system is maximised when $\sum_{i=1}^{P} X_i = 0$, which means that $Y_i = C$, so the maximum number of coins possible in the system is:

$$RB\sum_{i=1}^{P} C \tag{4}$$

### 4.3 Number of possible outcomes

The number of possible outcomes is crucial in any game as it dictates its magnitude and complexity of the game.

First, the total number of possibilities given $P$ players, $R$ rounds, and $C$ starting coins, and $B$ remaining constant, is calculated. On each round, each player has $C+1$ choices on how to allocate the coins. For every of $P_i$'s choice, $P_{i+1}$ can make $C+1$ choices, and $P_{i+2}$ can make $C+1$ choices, and so on. Thus we can express the number of possibilities in 1 round as:

$$(C+1)^P \tag{5}$$

From round 1, there is $(C + 1)^P$ possibilities, and in each of round 1's possibilities, there is $(C + 1)^P$, and for the next round, there is $(C + 1)^P$ possibilities of per possibility of the previous round, and so on. Hence we can express the number of possibilities as:

$$(C + 1)^{P \times R} \tag{6}$$

The issue with trying to find the Nash equilibrium by analysing all the possibilities is that there is simply too many combinations. Substituting the values, in 3.3 part 1(d), there are 28,430,288,029,929,701,376 combinations, which is too many for a typical computer to calculate.

However, as stated above, each player has $C + 1$ number of possible choices, which in game theory, correlates to the possible strategies. Each set of strategies (i.e. choices by $P_1$ to $P_i$) will only lead to one outcome in one round. Hence, the unique outcomes in one round can be calculated by finding the unique strategies.

## 4.4 Unique strategies

Let $S_{P,C}$ be the number of unique strategies for $P$ players and $C$ coins.

The number of unique strategies is dependent on two factors: the number of players, $P$, and the number of coins given to each player, $C$. To calculate the number of unique strategies, we can sort these set of strategies out. To illustrate this point, we would use $P = 5$ and $C = 5$. First, a combination of how many players pick the same strategy is made. In $P = 5$:

$$(1,1,1,1,1),(1,1,1,2),(1,1,3),(1,4),(1,2,2),(2,3),(5) \tag{7}$$

In the notation used above, each number represents the number of people who picked a specific strategy. For example, (5) meant that all 5 players picked the same strategy, and $(1,1,3)$, meant 3 strategies were chosen, and among the 5 players, 3 of them had the same strategy, while the other two had different strategies.

Next, since there is $C+1$ strategies, in this case 6, choose the number of strategies needed from 0 to $C$. For example, in (5), since only one strategy is used by this set of players, the number of strategies can be expressed as $\binom{6}{1}$. For $(1,2,2)$, 3 strategies are used, so it can be expressed $\binom{6}{3}$, and since 1 of the strategy is only picked by 1 player, we can express that strategy as $\binom{3}{1}$. Hence the total number of strategies for the set $(1,2,2)$ is $\binom{6}{3} \times \binom{3}{1}$.

Thus, the number of unique strategies for $P = 5$ and $C = 5$ is the sum of all the combinations listed in (7).

$$\begin{aligned} S_{5,5} &= \binom{6}{5} + \binom{6}{4}\binom{4}{1} + \binom{6}{3}\binom{3}{1} + \binom{6}{3}\binom{3}{1} + \binom{6}{2}\binom{2}{1} + \binom{6}{2}\binom{2}{1} + \binom{6}{1} \\ &= 252 \end{aligned} \tag{8}$$

And thus the number of unique outcomes derived from the number of unique strategies is:

$$(S_{P,C})^R \tag{9}$$

However, the problem is that $252^5 = 1,016,255,020,032$, which is significantly smaller than the total number of possible outcomes, but it is still too big for an average computer to process. To put this number into context, normal computers process 1 million commands in an average of 130 seconds. To computer 1,016,255,020,032 number of commands, it would take approximately 1500 days, which is 4.11 years!

Hence, this project's solution is to round up the number of coins that is redistributed to the players (see Equation (2)). Thus, the number of coins each player receives from the bank, after rounding, can be expressed as:

$$\lceil \frac{B \sum_{i=1}^{P} Y_i}{P} \rceil \tag{10}$$

## 4.5 Nash Equilibrium

Nash equilibrium, as defined in the literature review, can be simply defined as a strategy which a player does not stand to gain anything by deviating from it. In other words, it is the best possible strategy. The following nomenclature will be based on the normal form representation of a game.

The set of strategy for player $i$ is given as $S_i = \{0, ..., s_n, ..., C\}$, in which player $i$ can choose to play $s_j \in S_i$ where $Y_i = j$.

### 4.5.1 Standard public goods game

$B$ is taken as 2 for the following examples.

The logic behind the Nash equilibrium can be expressed by using a strategy-pay-off matrix. In this report, the format of the matrix will be as follows. The pay-off for player $i$ is denoted as $u_{p_i}$, and the set of strategies available to player $i$, $S_i$, contains the set of the number of coins the player will contributes to the bank. Since all players start with the same number of coins, the available strategies for each player is the same, i.e. for player 1 to player $i$, $S_1 = S_2 = ... = S_i$:

| | | Player 2 | |
|---|---|---|---|
| | | $s_1$ | $s_2$ |
| Player 1 | $s_1$ | $u_{p_1}, u_{p_2}$ | $u_{p_1}, u_{p_2}$ |
| | $s_2$ | $u_{p_1}, u_{p_2}$ | $u_{p_1}, u_{p_2}$ |

Table 1: Format of strategy-pay-off matrix

To start out, the Nash equilibrium for $P = 2$ and $C = 1$ is investigated. As such, the strategies available for both players is expressed as $S = \{0, 1\}$. The payoff is calculated using the equation in Equation (3). The strategy-pay-off matrix is as follows:

| | | Player 2 | |
|---|---|---|---|
| | | 0 | 1 |
| Player 1 | 0 | $1, 1$ | $2, 1$ |
| | 1 | $1, 2$ | $2, 2$ |

Table 2: Strategy-pay-off matrix for $P = 2$, $C = 1$

Let $(q, 1 - q)$ be the probability of any player $i$ of playing $S = \{0\}$ and $S = \{1\}$ respectively where $0 \leq q \leq 1$. Player $i$'s expected pay-off for playing strategy $s_k$ will be the sum of the product of pay-off of $s_k$ against strategies $s_1$ to $s_n$ of the other player(s). Thus, player 1's expected pay-off, $ep_i$, for playing $S = \{0\}$ is:

$$
\begin{aligned}
ep_0 &= q \times 1 + q \times 2 \\
&= 3q \qquad (11)
\end{aligned}
$$

and player 1's expected pay-off for playing $S = \{1\}$ is:

$$
\begin{aligned}
ep_1 &= (1-q) \times 1 + (1-q) \times 2 \\
&= 3(1-q) \\
&= 3 - 3q \qquad (12)
\end{aligned}
$$

Each player aims to maximise their utility. Hence, to maximise Player 1's utility for choosing strategy $S = \{0\}$ and $S = \{1\}$, $ep_0$ and $ep_1$ has to be maximised under the constraints $0 \leq q \leq 1$. For equation (11), $3q$ is maximised when $q = 1$ and for equation (12), $3 - 3q$ is maximised when $q = 0$. Hence, for the set of strategy $S = \{0, 1\}$, the probability of $S = \{0\}$ being played is 1 and $S = \{1\}$ is 0, making $S = \{0\}$ the Nash equilibrium optimal strategy.

As mentioned, each player has two strategies available: to contribute no coins ($Y = 0$), or to contribute all their coins ($Y = 1$). Both players' pay-off is maximised when they both contribute 1 coin, where their final pay-off is $(2, 2)$. However, if Player 1 is going to contribute 0 coins, Player 2 would also want to contribute 0 coins to arrive at the pay-off of $(1, 1)$ rather than losing to Player 1 by contributing 1 coin, getting the pay-off $(2, 1)$. Similarly, if Player 1 were to contribute all the coins, Player 2 would want to contribute 0 as that would lead to Player 2 winning Player 1 with the pay-off $(1, 2)$ rather than contributing 1 coin and arrive at a draw with Player 1, at $(2, 2)$, and vice versa. Therefore, $S = \{1\}$ (contributing 1 coin) is strictly dominated by $S = \{0\}$ (contributing 0 coins), resulting in $S = \{0\}$ being the Nash equilibrium optimal strategy. In other words, you cannot lose by contributing nothing.

What if $P > 2$ or $C > 1$? Will the Nash equilibrium be the same? To do so, the strategy-pay-off matrix can be created. $P = 2$ and $C = 5$ is defined in the following example:

Player 2

| | | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| | 0 | 5,5 | 6,5 | 7,5 | 8,5 | 9,5 | 10,5 |
| | 1 | 5,6 | 6,6 | 7,6 | 8,6 | 9,6 | 10,6 |
| Player 1 | 2 | 5,7 | 6,7 | 7,7 | 8,7 | 9,7 | 10,7 |
| | 3 | 5,8 | 6,8 | 7,8 | 8,8 | 9,8 | 10,8 |
| | 4 | 5,9 | 6,9 | 7,9 | 8,9 | 9,9 | 10,9 |
| | 5 | 5,10 | 6,10 | 7,10 | 8,10 | 9,10 | 10,10 |

Table 3: Strategy-pay-off matrix for $P = 2$, $C = 5$

The logic for this extended version of the game is the same as the example before. The expected pay-off for player $i$'s choice of strategies $S = \{0\}$ and $S = \{5\}$, where $(q, 1 - q)$ is the probability distribution for $S = \{0\}$ and $S = \{5\}$, is $ep_0 = 45q$ and $ep_5 = 45 - 45q$ respectively. Maximizing the two equations, $q = 1$ for $S = \{0\}$ and $q = 0$ for $S = \{5\}$, making $S = \{0\}$ still the Nash equilibrium.

The value of $C$ does not affect the Nash equilibrium, as seen in table 3. The pay-off can be interpreted as a draw (denoted by 0), Player 1 wins (denoted by 1) and Player 2 wins (denoted by 2):

| | | Player 2 | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 | 4 | 5 |
| | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 2 | 0 | 1 | 1 | 1 | 1 |
| Player 1 | 2 | 2 | 2 | 0 | 1 | 1 | 1 |
| | 3 | 2 | 2 | 2 | 0 | 1 | 1 |
| | 4 | 2 | 2 | 2 | 2 | 0 | 1 |
| | 5 | 2 | 2 | 2 | 2 | 2 | 0 |

Table 4: Strategy-result matrix for $P = 2$ and $C = 5$

As seen in Table 4, Player 1 wins when $S_1 < S_2$, and Player 2 wins when $S_2 < S_1$, and a draw occurs when $S_1 = S_2$. Hence, the Nash equilibrium of the game is not affected by the value of $C$ as the most optimal strategy is to contribute the minimum possible amount, which is 0.

How does the $P$ affect the Nash equilibrium? In Equation (3), the number of coins a player gets in a round is $\frac{B \sum_{i=1}^{P} Y_i}{P} + X_i$, and a player wins when the value the above equation is maximised. The first term is a constant as every player gets the same number of coins when it is redistributed from the bank, and hence it is the second term, $X_i$, which is needed to be maximised to determine if a player wins. $X_i$ is maximised when a player contributes 0 coins to the bank as $C = X_i + Y_i$, given $C = 5$, when the contribution to the bank of Player $i$, $Y_i$ is 0, $X_i = 5 - 0$ which is maximised. Thus the number of players does not affect the Nash equilibrium.

Therefore, a computer program was written using the algorithm shown in Equation (3) and section 4.4 to generate all 252 combinations of the strategies in $R = 1$, and based on Equation (10) to generate all the 584,760 unique outcomes when $R = 5$ (based on looping the 252 strategies $R$ times) after the coins that were redistributed were rounded, where $P = 5$ and $C = 5$. The data was sorted to allow it to be easily analysed by arranging the contributions of players $P_1$ to $P_5$ where $Y_{P_1} \leq Y_{P_2} \leq Y_{P_3} \leq Y_{P_4} \leq Y_{P_5}$. The results of the analysis of both sets of data are shown below.

| | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ |
|---|---|---|---|---|---|
| Winrate | 50.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Drawrate | 50.00 | 50.00 | 22.22 | 8.33 | 2.38 |
| Loserate | 0.00 | 50.00 | 77.78 | 91.67 | 97.62 |

Table 5: Results after 1 round where $P = 5$ and $C = 5$

| | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ |
|---|---|---|---|---|---|
| Winrate | 83.846 | 0.000 | 0.000 | 0.000 | 0.000 |
| Drawrate | 16.154 | 16.154 | 1.986 | 0.180 | 0.004 |
| Loserate | 0.000 | 83.864 | 98.014 | 99.820 | 99.996 |

Table 6: Results after 5 rounds with rounding off where $P = 5$ and $C = 5$

As shown in the data, as $P_1$'s contribution is the lowest among the 5 players, the worst result for $P_1$ is a draw and otherwise, it is a win, supporting that the Nash equilibrium is to contribute the lowest possible amount, 0.

Furthermore, a winrate vs contribution graph can be plotted to examine the relationship between the two variables.

A best fit exponential function is generated for the above graph,

$$y = 84.3e^{0.209x} \tag{13}$$

with a $R^2$ value of 0.988. As seen from the graph and equation, the greater the contribution, the lower the winrate.

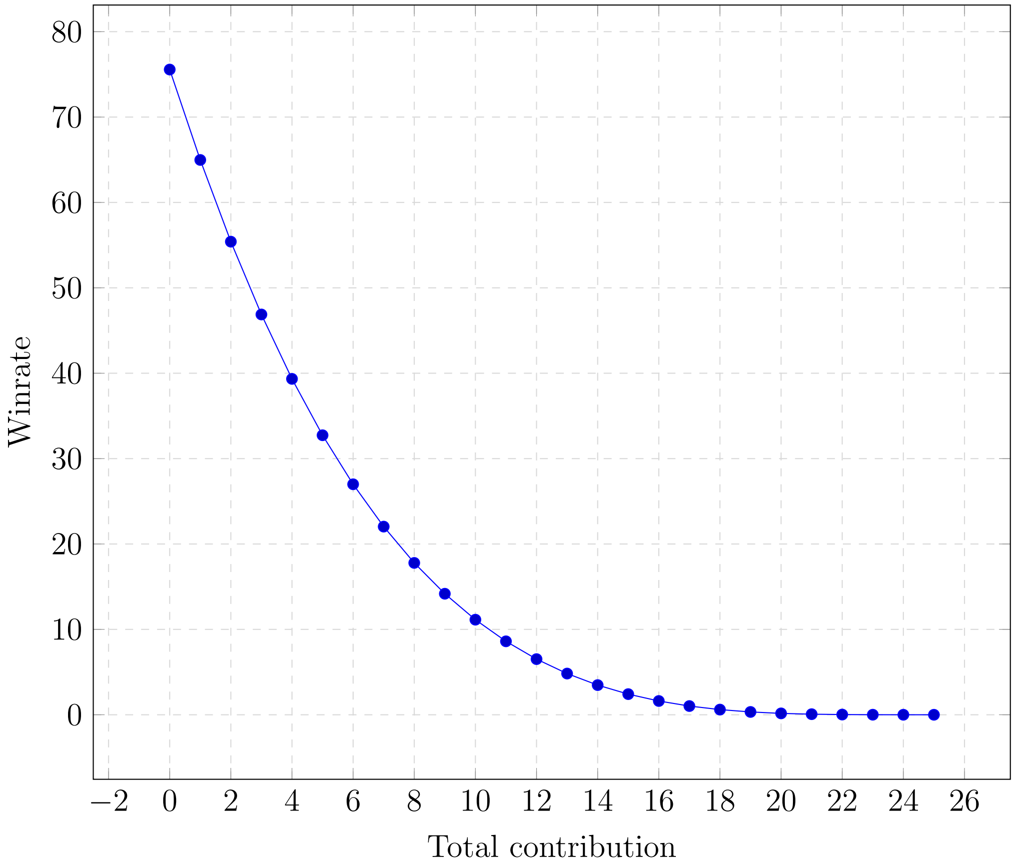


Figure 1: Winrate vs total contribution

### 4.5.2 Public goods game with a win condition

In this variation of the public goods game, the players have to reach a minimum number of coins, $M$, after $R$ rounds to be eligible to win the game. For $P = 5$, $C = 5$ and $R = 5$, $M = 40$, but how is $M$ decided?

Figure 1 is a frequency vs final score graph generated from the data described in 4.5.1, showing the results of all 584,670 combinations where the scores have been rounded. The graph resembles a bell curve.

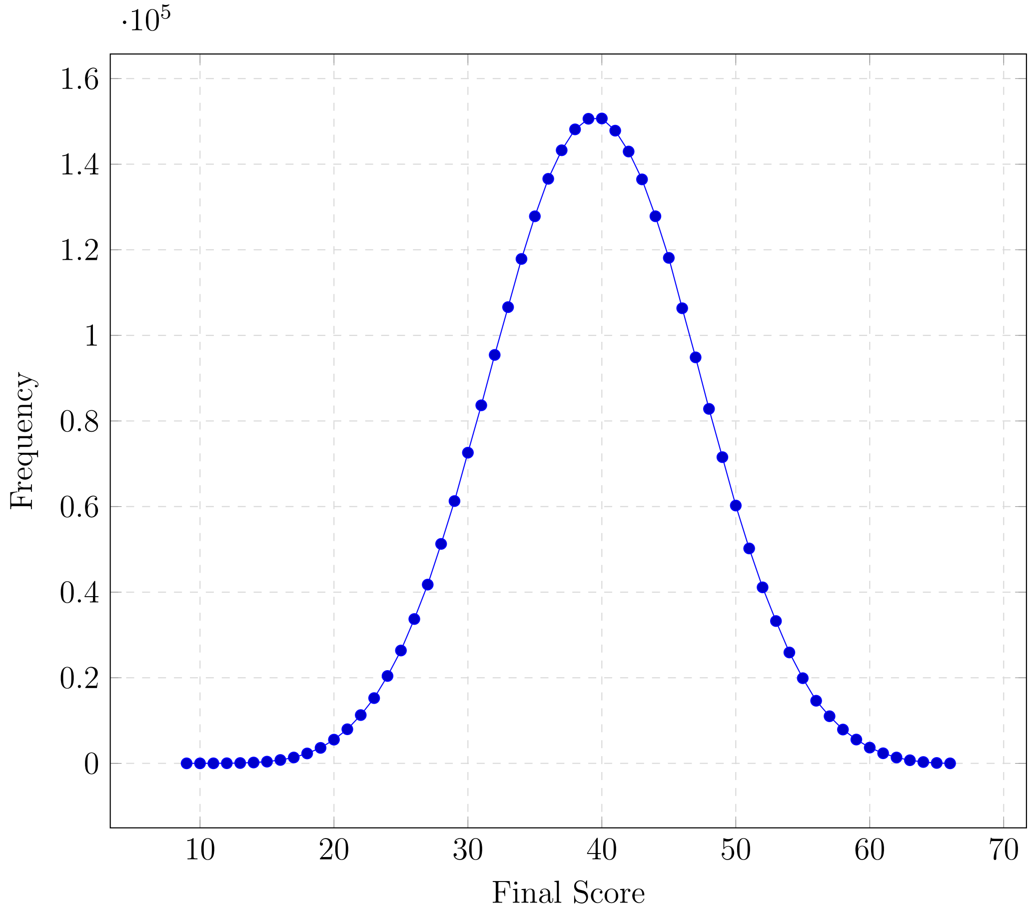


Figure 2: Frequency vs Final Score

| | |
|---|---|
| Average Final Score | 39.4665 |
| Median Final Score | 39 |
| Score of minimum 39/% | 55.0058 |
| Score of minimum 40/% | 49.8547 |

Table 7: Average, median, and probability of getting a score of 39 and 40

Further analysis of the data shows that the average final score is 39.4665, and the median score is 39. $M$ is determine as a cut-off point to eliminate the players who get a below average final score. 40 was chosen instead of 39 is because approximately 50% of the players will get 40 coins, compared to

55% of 39.

Before determining the Nash equilibrium for this variation of the public goods game, the probability of each player achieving a final score of 40 is analysed, based on the conditions defined above, $Y_{P_1} \leq Y_{P_2} \leq Y_{P_3} \leq Y_{P_4} \leq Y_{P_5}$.

| | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ |
|---|---|---|---|---|---|
| Success | 92.895 | 83.454 | 49.847 | 16.228 | 6.581 |
| Failure | 7.105 | 16.546 | 50.103 | 83.772 | 93.149 |

Table 8: Probability of getting a final score of 40 based on strategy

As seen in table 8, a player will lose if he/she fails to get 40 coins; unlike the standard public goods game, it is possible to lose by contributing nothing. Hence, what is the Nash equilibrium for this variation? To do so, the strategy-pay-off matrix can be used. As we have established, $P$ and $C$ does not affect the Nash equilibrium. Thus, we will only consider the extremes of the players available strategies and a random strategy between the two extremes : $S = \{0, 3R, C \times R\}$ for $P = 2$, $C = 5$ and $R = 5$.

| | | Player 2: 0 | 15 | 25 |
|---|---|---|---|---|
| Player 1 | 0 | 25,25 | 40,25 | 50,25 |
| | 15 | 25,40 | 45,45 | 55,40 |
| | 25 | 25,50 | 40,55 | 50,50 |

Table 9: Strategy-pay-off matrix for $P = 2$, $C = 5$ and $R = 5$ with win condition of 40

Afterwards, convert the strategy-pay-off matrix to the strategy-result matrix where -1 denotes a loss for both players, 0 denotes a draw, 1 denotes win for Player 1 and 2 denotes win for Player 2:

Player 2

| | | 0 | 15 | 25 |
|---|---|---|---|---|
| Player 1 | 0 | -1 | 1 | 1 |
| | 15 | 2 | 0 | 1 |
| | 25 | 2 | 2 | 0 |

Table 10: Strategy-result matrix for $P = 2$, $C = 5$ and $R = 5$ with win condition of 40

When both players contribute 0, they end with 25 coins which does not meet the minimum 40 coins, and thus they both lose. However, one player can only stand to win the other by contributing less than the other player, otherwise it will end in a draw or loss (because they did not meet the minimum 40 condition). In other words, by contributing 0, the player will only lose to the minimum 40 condition and not to other players, whereas by contributing more than other players, not only will that player lose to other players, but also stand a chance to not meet the minimum 40 condition. Thus, the Nash equilibrium is still to contribute 0, $S = \{0\}$, albeit standing a chance to lose the game.

This can be proven by calculating the estimated pay-off for a player. In this case, the estimated pay-off player 1 is calculated. -2 is used to denote a loss the other player, -1 is used to denote a loss from the win condition only and a draw with the other player, 0 is used to denote a draw for both players who reached the win condition, and 1 denotes a win. The pay-off is not arbitrarily decided, but rather based on how good the result is to the player. Losing to the other player is worse than winning/drawing with other players but losing to the win condition, which is why the pay-off for both losses are different. Let $(q, r, 1-q-r)$ be the probability distribution for the strategies

$S = \{0, 15, 25\}$ where $0 \leq q, r, 1 - q - r \leq 1$ and $q + r + (1 - q - r) = 1$.

Player 2

| | | 0 | 15 | 25 |
|---|---|---|---|---|
| Player 1 | 0 | -1 | 1 | 1 |
| | 15 | -2 | 0 | 1 |
| | 25 | -2 | -2 | 0 |

Table 11: Strategy-result matrix for $P = 2$, $C = 5$ and $R = 5$ with win condition of 40 from player 1's perspective

Thus the expected pay-off of only $S = \{0\}$ and $S = \{15\}$ needs to be calculated to find the value of $q$ and $r$.

$$\begin{aligned} ep_0 &= -1 \times q + q + q \\ &= q \end{aligned} \tag{14}$$

$$\begin{aligned} ep_{15} &= -2 \times r + r \\ &= -r \end{aligned} \tag{15}$$

Equation 14 is maximised when $q = 1$ and equation 15 is maximised when $r = 0$. Thus, the Nash equilibrium is still $S = \{0\}$.

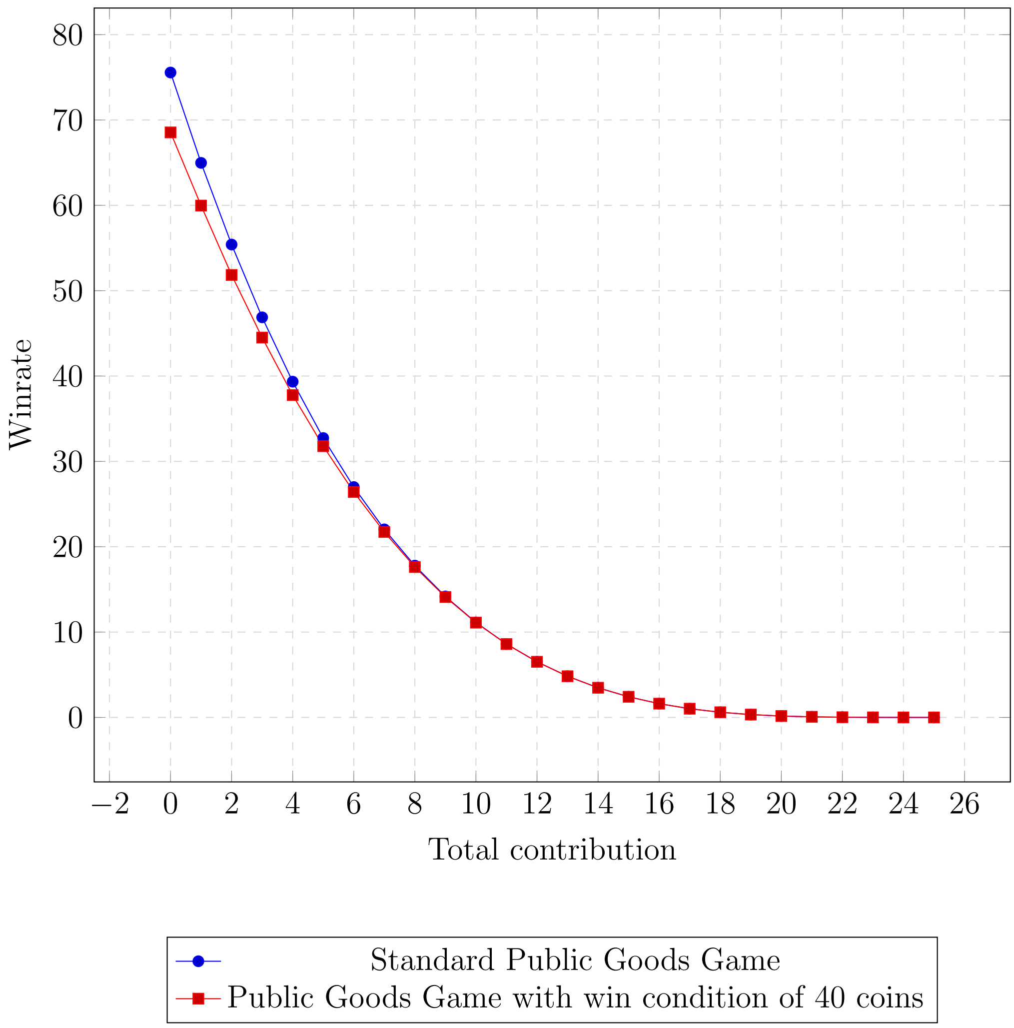


Figure 3: Winrate vs total contribution between the standard public goods game and the public goods game with the aforementioned win condition

The red line could be written as

$$y = 77.9e^{-0.202x} \tag{16}$$

The loserate vs total contribution graph can be plotted to examine the relationship between the two variables.

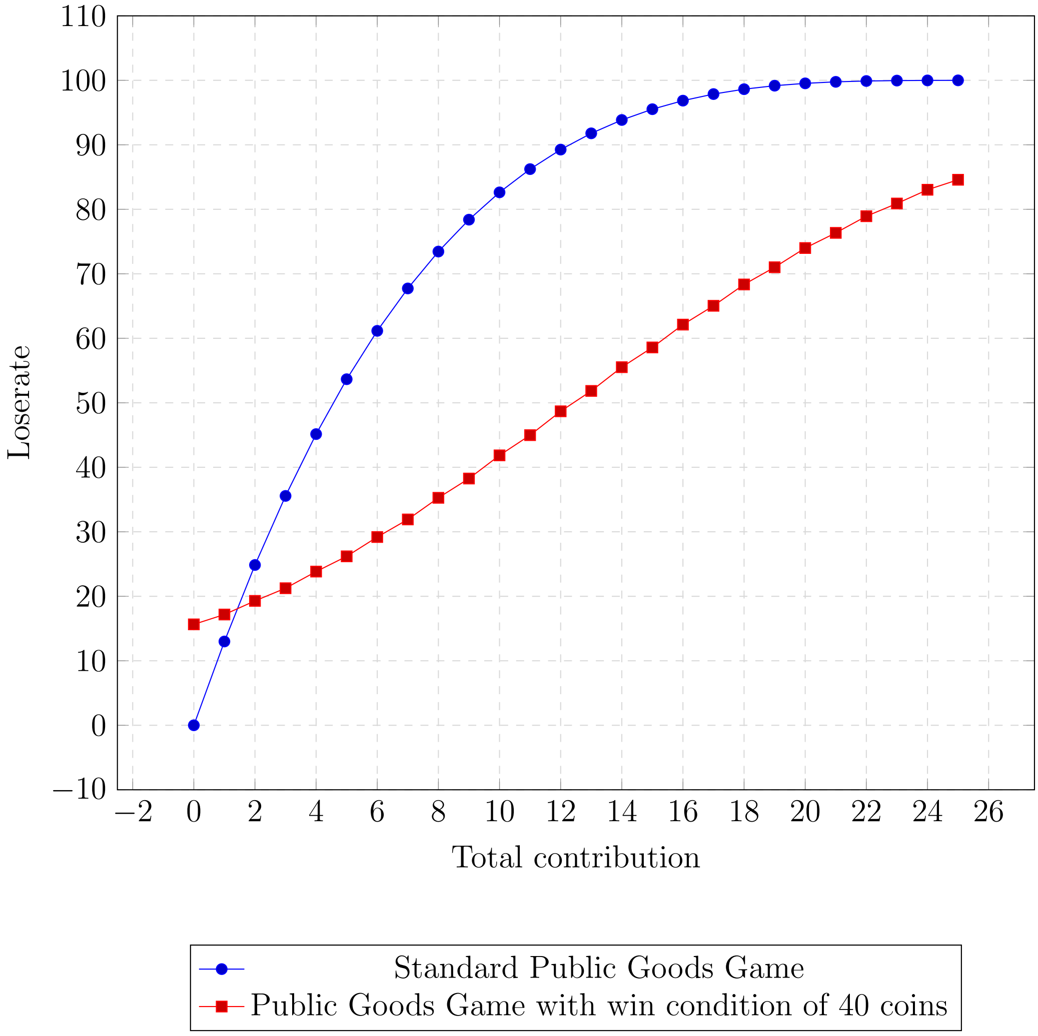


Figure 4: Loserate vs total contribution between the standard public goods game and the public goods game with the aforementioned win condition

As expected, when a win condition is imposed, the winrate of contributing the minimum possible amount decreases, and, by following the Nash equilibrium optimal strategy, one has the highest chance to win while having the lowest chance to lose.

### 4.5.3 Public goods game with penalty towards the free-loader

In this variation of the public goods game, the free-loader is punished at the end of the round. The free-loader is the player who contributes the least, and therefore have the highest score. Thus, to discourage such behaviour, at the end of the game, the player with the highest score (aka the free-loader) is kicked from the match, and the next player wins. If multiple players are in first place, all of them are kicked and thus they lose.

As such, to maximise one's win to lose ratio, one should always aim to be the only second place player for a win, or a tied second place for a draw. But how exactly can a player aim for second place when he/she does not know what other players are contributing?

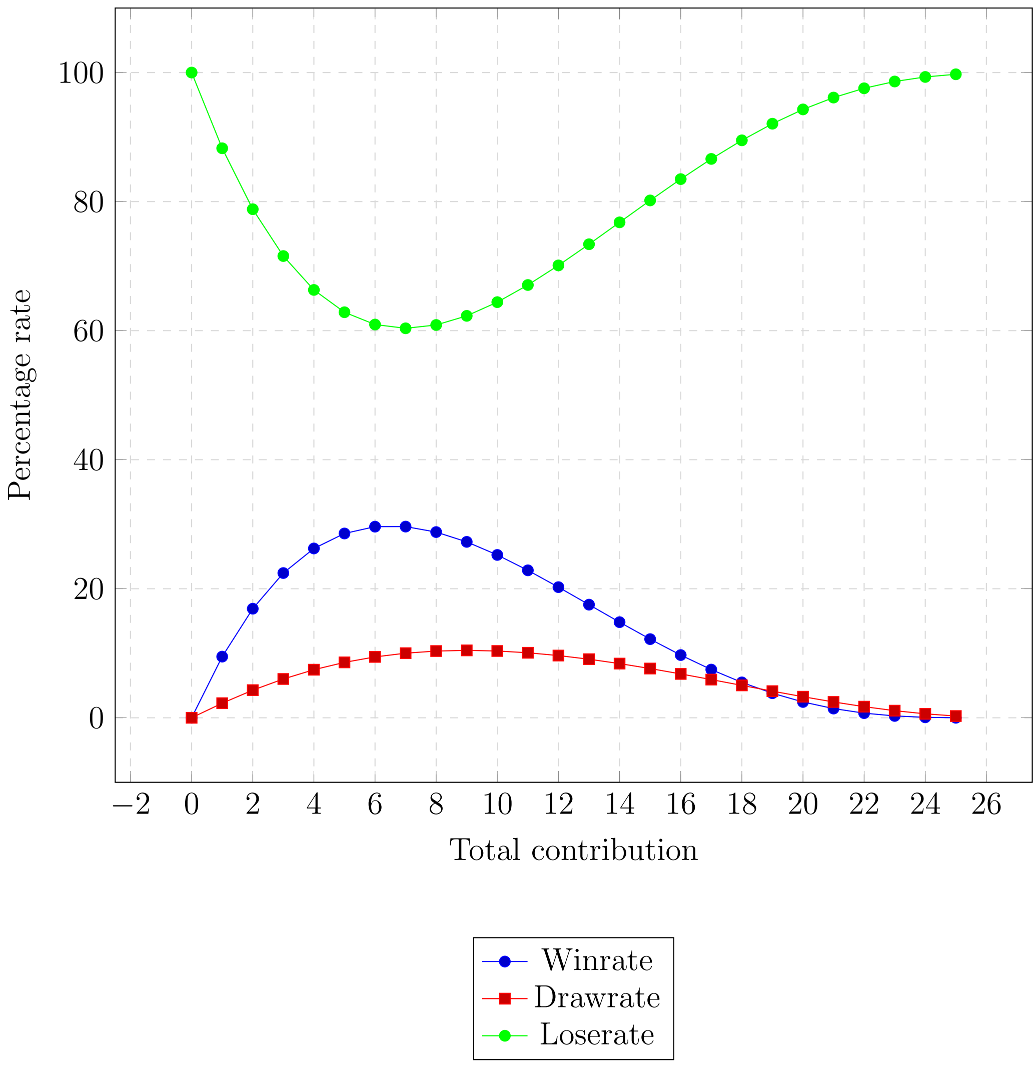


Figure 5: Winrate, drawrate and loserate vs total contribution

As seen in Figure 5, the winrate to lose rate ratio is maximised when total contribution is equals to 7. This means that statistically, you will have the highest chance of winning and lowest chance of losing by contributing 7 coins. However, this is not the Nash equilibrium optimal strategy because if one player were to contribute 7 regardless of what other players choose, and the remaining four players contribute more than 7, this player will end up as first place and thus lose the game.

To visualise how this variation of the game will play out, a multi-dimensional strategy-pay-off matrix can be used, and then converted to a strategy-result matrix For simplicity, let $P = 3$ and $C = 3$. Pay-off was rounded off to 1 decimal place. The following 4 matrices will use the following abbreviations: $L$ to denote all players lost, $D(x, y)$ to denote players $x$ and $y$ have drew and $x$ to denote player $x$ has won.

$P_1$ plays 0 only.

| $P_2$ \ $P_3$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $L$ | 3 | 3 | 3 |
| 1 | 2 | $D(2,3)$ | 2 | 2 |
| 2 | 2 | 3 | $D(2,3)$ | 2 |
| 3 | 2 | 3 | 3 | $D(2,3)$ |

Table 12: Strategy-result matrix for $P_2$ and $P_3$ when $P_1$ plays 0

$P_1$ plays 1 only.

| $P_2$ \ $P_3$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | $D(1,3)$ | 1 | 1 |
| 1 | $D(1,2)$ | $L$ | 3 | 3 |
| 2 | 1 | 2 | $D(2,3)$ | 2 |
| 3 | 1 | 2 | 3 | $D(2,3)$ |

Table 13: Strategy-result matrix for $P_2$ and $P_3$ when $P_1$ plays 1

$P_1$ plays 2 only.

$P_3$

| | | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $P_2$ | 0 | 1 | 3 | $D(1,3)$ | 1 |
| | 1 | 2 | 1 | $D(1,3)$ | 1 |
| | 2 | $D(1,2)$ | $D(1,2)$ | $L$ | 3 |
| | 3 | 1 | 1 | 2 | $D(2,3)$ |

Table 14: Strategy-result matrix for $P_2$ and $P_3$ when $P_1$ plays 2

$P_1$ plays 3 only.

$P_3$

| | | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $P_2$ | 0 | 1 | 3 | 3 | $D(1,3)$ |
| | 1 | 2 | 1 | 3 | $D(1,3)$ |
| | 2 | 2 | 2 | 1 | $D(1,3)$ |
| | 3 | $D(1,2)$ | $D(1,2)$ | $D(1,2)$ | $L$ |

Table 15: Strategy-result matrix for $P_2$ and $P_3$ when $P_1$ plays 2

As seen from Tables 12 to 15, no players will play $S = \{0\}$ as it will always guarantee them a first place, and thus resulting in a loss for them. This makes $S = \{0\}$ a strictly dominated strategy, and hence no rational players would play it. Once this is established, the next minimum number of coins a player can play is $S = \{1\}$. But since all players will not play $S = \{0\}$, $S = \{1\}$ will guarantee the player a first place, making it a strictly dominated strategy, and this process is iterated for $S = \{2, ..., C - 1\}$, until the contribution cannot be increased any further, making $S = \{C\}$ the only non-strictly dominated strategy, thus $S = \{C\}$ is the Nash equilibrium.

To prove this, the expected pay-off of a player can be calculated. In this case, the expected pay-off for player 2 shall be calculated. Once again, the pay-off has to be redefined from the perspective of player 2. -2 would mean that player 2 has lost to other players by either coming in as 1st place or coming in after 2nd place, -1 would denote that player 2 has drawn the 1st place with all other players, 0 would mean that player 2 has drawn 2nd place with other players, and 1 would mean that player 2 has won by being the only second place. Once again, the pay-off is decided how good or bad the result is to the player. Let $(q_0, q_1, q_2, 1 - q_0 - q_1 - q_2)$ be the probability distribution of player 2 playing strategies $S = \{0, 1, 2, 3\}$:

$$\begin{aligned} ep_0 &= 12(-2 \times q_0) \\ &= -24q_0 \qquad (17) \\ ep_1 &= 8(-2 \times q_1) - 1 \times q_1 + 0 \times q_1 + 2 \times q_1 \\ &= -15q_1 \qquad (18) \\ ep_2 &= 4(-2 \times q_2) - 1 \times q_2 + 3(0 \times q_2) + 4 \times q_2 \\ &= -5q_2 \qquad (19) \end{aligned}$$

In all cases, equations (17) to (19) will be maximised when $q_0, q_1, q_2 = 0$, thus the probability a player should play $S = \{3\}$ is $1 - 0 - 0 - 0 = 1$, making $S = \{3\}$ the Nash equilibrium optimal strategy.

## 4.6 Empirical experiment

The public goods game with minimum 40 coins win condition and the public goods game with a penalty towards the free-loader are played 5 times respectively. The following results is limited to the small sample size we have ($2 \times 5 \times 5 = 50$ sets of data). This is because we could only conduct our experiment online with people we know due to the Covid-19 measures. All the games are conducted with $P = 5$, $C = 5$, $R = 5$ and $B = 2$.

### 4.6.1 Results for public goods with minimum 40 coins

As established above in section 4.5, the Nash equilibrium for this game is 0. Among the 25 players in our experiment, 12 of them followed the Nash equilibrium. Among these 12 players, 2 of them won, 5 of them drew, and the remaining 5 lost because they did not reach the minimum 40 coins. The rest of the players did not win at all.

The number players who played the antithesis of the Nash equilibrium strategy - by contributing 25 coins - is 3. And all of them lost.

The following table shows other information which is interesting to compare between this current variation and the following variation of the game.

| | |
|---|---|
| Number of players who played the Nash equilibrium | 12 |
| Number of players who played the worst strategy | 3 |
| Average contribution in 1 round | 1.568 |
| Median contribution in 1 round | 0 |
| Mode contribution in 1 round | 0 |
| Average final score | 34.96 |
| Median final score | 35 |
| Number of players with minimum 40 coins | 10 |
| Average score of 1st place | 42.8 |
| Average score of 2nd place | 29.8 |
| Average score of last place | 23.8 |

Table 16: Results of the public goods with minimum 40 coins condition

### 4.6.2 Results for public goods game with penalty

The results in this variation is the extremely interesting. Among the 6 players who have played the Nash equilibrium optimal strategy where $Y = C$, 3 players in one game drew, and 3 other players in 3 separate games who played the Nash equilibrium lost. This is because other players did not play rationally, in which if they all did, all of them would end up drawing in first place. Unlike the results of the game with the minimum 40 strategy, where the median and mode of the players' choices are the Nash equilibrium, most players did not follow the Nash equilibrium in this variation.

With the introduction of a penalty, the average contribution also went up, as seen from the results in table 16.

| | |
|---|---|
| Number of players who played the Nash equilibrium | 6 |
| Number of players who played the worst strategy | 0 |
| Average contribution in 1 rounds | 3.12 |
| Median contribution in 1 round | 3 |
| Mode contribution in 1 round | 2 |
| Average final score | 42.40 |
| Median final score | 42 |
| Number of players with minimum 40 coins | 18 |
| Average score of 1st place | 48.4 |
| Average score of 2nd place | 47.2 |
| Average score of last place | 35 |

Table 17: Results of the public goods with the introduction of penalty

## 4.7 Comparison of the results of the empirical experiment

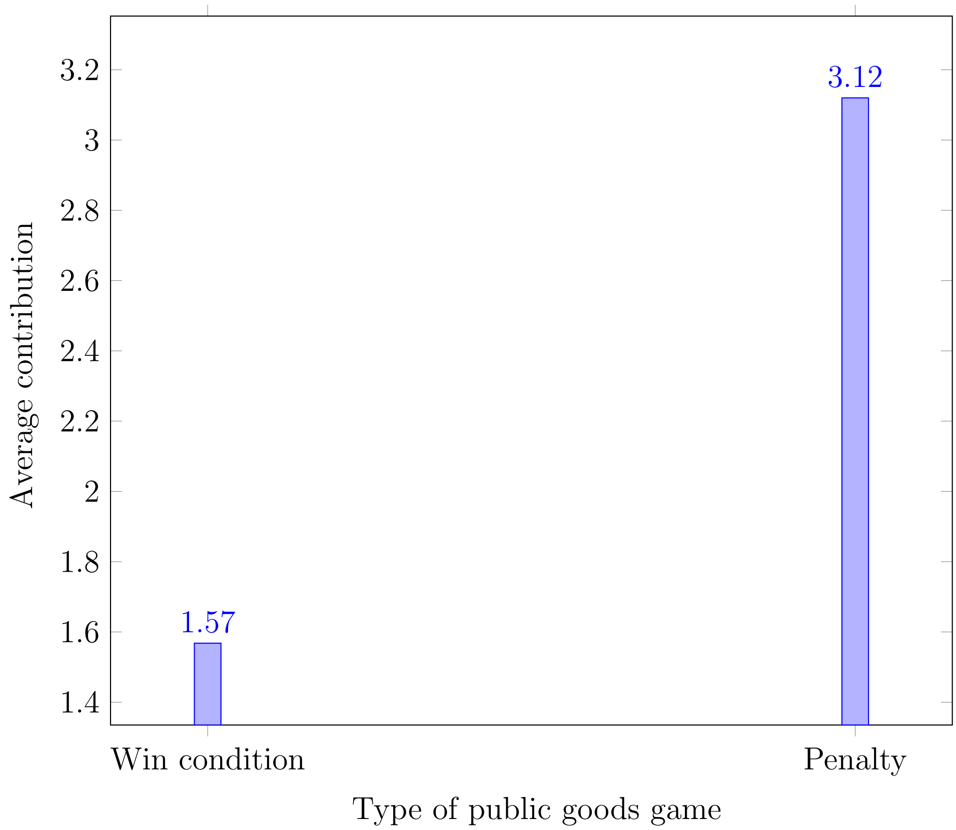


Figure 6: Average contribution vs Type of public goods game

As seen in figure 6, the average contribution per player per round when there is a penalty is nearly twice of that in the win condition.

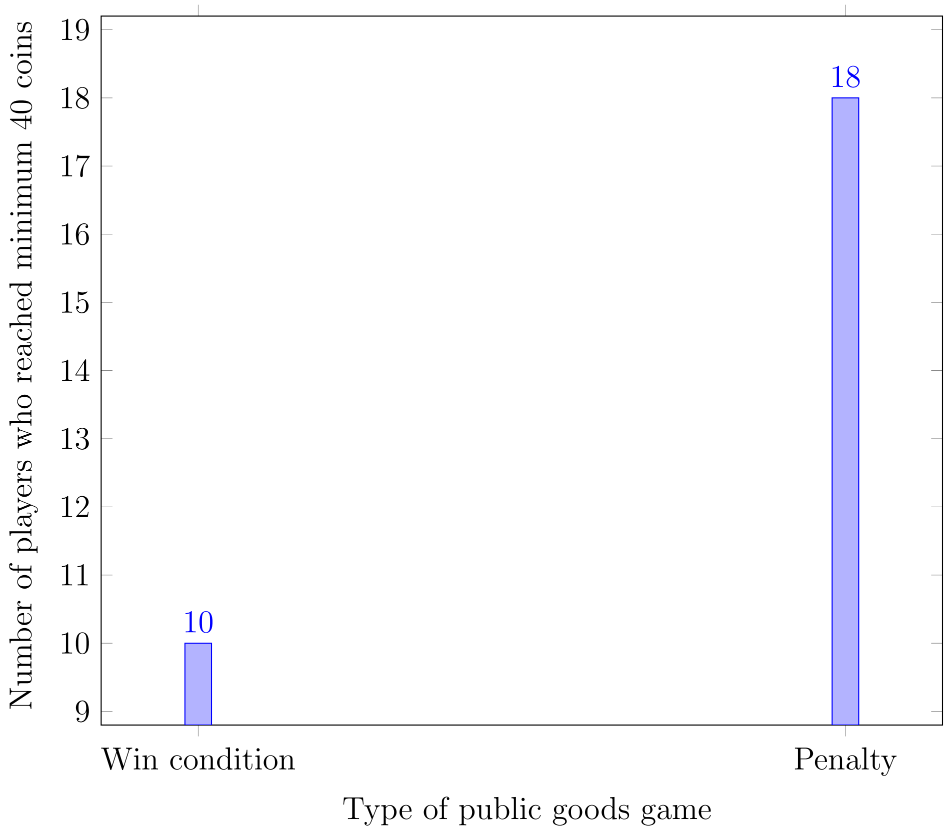


Figure 7: Average contribution vs Type of public goods game

As a result of the players each contributing more, more players would be able to reach 40 coins with the penalty than the win condition, even though the win condition requires the players to reach 40 coins while the penalty variation did not.

# 5 Conclusion

### Theory

1. The Nash equilibrium for the standard public goods game is to con-

tribute 0.

2. With the addition of a win condition in the standard public goods game, the Nash equilibrium remains at 0.

3. With the addition of a penalty to the free-loader, the Nash equilibrium for the public goods game is to contribute the maximum possible number of coins.

**Empirical**

1. Players who played according to the Nash equilibrium for the public goods game with win condition wins the most.

2. Players who played according to the Nash equilibrium for the public goods game with penalty either lost or drew because other players were not playing rationally.

3. The introduction of a penalty was more effective in raising contributions than the win condition does.

# 6 Future work

1. To explore the the Nash equilibrium of other variations of the public goods game, such as the open public goods game, public goods game with rewards, income variation, et cetera.

2. Increase the sample size of the empirical experiment.

# 7 References


1. Andreoni, J. (1998). Toward a Theory of Charitable Fund-Raising. Journal of Political Economy, 106(6), 1186–1213. doi: 10.1086/250044

2. Cowen, T. (1985). Public Goods Definitions and their Institutional Context: a Critique of Public Goods Theory. Review of Social Economy, 43(1), 53–63. doi: 10.1080/00346768500000020

3. Ferber, R., Luce, R. D., & Raiffa, H. (1959). Games and Decisions: Introduction and Critical Survey. Journal of Marketing, 23(4), 469. doi: 10.2307/1247425

4. Gibbons, R. (2013). A primer in game theory. Prentice Hall.

5. Goeree, J. K., & Holt, C. A. (2005). An Explanation of Anomalous Behavior in Models of Political Participation. American Political Science Review, 99(2), 201–213. doi: 10.1017/s0003055405051609

6. Hasson, R., Löfgren, Å., & Visser, M. (2010). Climate change in a public goods game: Investment decision in mitigation versus adaptation. Ecological Economics, 70(2), 331–338. doi: 10.1016/j.ecolecon.2010.09.004

7. Leshem, A., & Zehavi, E. (2008). Cooperative Game Theory and the Gaussian Interference Channel. IEEE Journal on Selected Areas in Communications, 26(7), 1078–1088. doi: 10.1109/jsac.2008.080906

8. George, C. (n.d.). Some Notation and Definitions. Retrieved August 10, 2020, from `https://academics.hamilton.edu/economics/cgeorges/game-theory-files/Notation-Definitions.pdf`

9. Mailath, G. (1998). Do People Play Nash Equilibrium? Lessons from Evolutionary Game Theory. Journal of Economic Literature, 36(3), 1347-1374. Retrieved March 17, 2020, from www.jstor.org/stable/2564802

10. McNulty, D. (2020, January 29). The Basics Of Game Theory. Retrieved March 16, 2020, from https://www.investopedia.com/articles/financial-theory/08/game-theory-basics.asp

11. Public goods game. (2020, January 10). Retrieved March 16, 2020, from https://en.wikipedia.org/wiki/Public_goods_game