https://sites.google.com/hci.edu.sg/hs-pw/home/project-categories/cat-8-mathematics?authuser=0

1. Introduction

When we started this project, we were intrigued with the pyramids of ancient egypt, and in particular, we wanted to see the structure of the pyramid, built it without modern day calculators or measurements, and we wanted to examine the construction of the pyramids, mainly the king's chamber and how it is constructed in relation to the rest of the pyramids.

Hence we started our project with the objective of exploring the construction of the king's chamber in relation to the pyramids they were constructed in, mainly how big could the king's chamber be in different constructions of pyramids, factoring in pressure and structural strength.

Hence we formulated the following research questions:

1. What is the relationship between the dimensions of the pyramid and the size of the king's chamber?
2. How does varying n, the number of sides of the pyramid affect the size of the king's chamber?
3. Given the size, height and angle of a pyramid, is it possible to calculate the largest king's chamber of any shape that can fit inside the pyramid?

In our study, we realised that there were a multitude of factors that could affect the construction of the pyramids, hence we assumed that in our constructions there would not be any external factors such as wing or rain, and that the pyramid is constructed from a single piece of sandstone and not many blocks of sandstone, as the gaps between the blocks of sandstone would affect our construction. We lastly assumed that sandstone weighed 2323 kg/m^3, and that the structural strength of sandstone was 55Mpa/ 5500000 kg/m^2

2. Literature review

The pyramids in Egypt are one of the most prominent examples of ancient engineering and mathematics in the world that we live in. The pyramids were originally built by the pharaohs of ancient egypt as a tomb for when they died. Representing the rays of the sun, it was thought that the pyramid could launch the deceased pharaoh's soul to be among the gods in the sky.

There are about 120 such structures, the biggest of which is the great pyramid of giza (also known as the great pyramid of khufu),standing at 139 meters tall and having a square base of length 230 meters

In every pyramid, there was always a king's chamber. The king's chamber is an empty, hollow space on the inside of the pyramid which contains the king's sarcophagus, and is the king's final resting ground. It is surrounded by walls made of solid granite, and hieroglyphs depicting history and culture are written onto these walls.

There is currently no known link between the size of the pyramid and the size of the king's chamber as all are about the same size, or what is the optimal shape of kings chamber or shape of the pyramid to maximise the size of the kings chamber, hence we would like to find out more in these aspects, so as to deepen our understanding of such ancient marvels

resources:

The Editors of Encyclopaedia Britannica. (2020, February 20). Pyramid. Retrieved August 17, 2020, from https://www.britannica.com/technology/pyramid-architecture

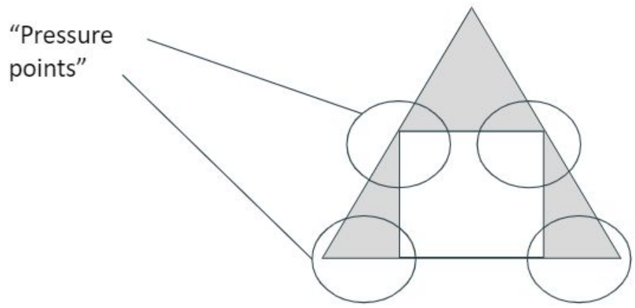
THE SPIRITUAL SIGNIFICANCE OF THE EGYPTIAN PYRAMIDS. (2014, May 28). Retrieved August 17, 2020, from https://jbhengu.wordpress.com/2014/04/23/the-spiritual-significance-of-the-great-pyramids-of-egypt/

Craig Freudenrich, P. (2020, April 02). How Pyramids Work. Retrieved August 17, 2020, from https://science.howstuffworks.com/engineering/structural/pyramid2.htm

3. The study and methodology

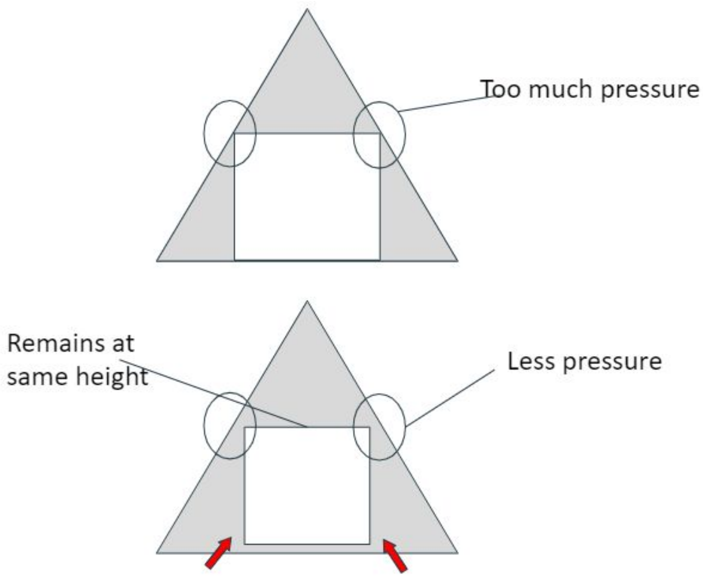
RQ1 What is the relationship between the dimension of a cubic king's chamber and the size of the pyramid?

In order to solve this question we made the assumption that the king's chamber was a cube as we did not want to calculate the different types of cuboids that can be used as the shape of the king's chamber as this would make our question extremely complex.



Cross section of pyramid

We can see that most of the pressure from the weight of the pyramid will be concentrated at the bottom of the pyramid and at the top of the king's chamber(as seen in diagram above). So, to arrive at a solution,we should calculate the pressure on these "pressure points", and by the amount of pressure they can hold, find the maximum possible size of the king's chamber based on the dimensions of the pyramid.



Our method was to first find the maximum size of the cube that can fit inside, then shrink the cube from the bottom such that the top of the king's chamber remains at the same height. This will allow the amount of sandstone above the king's chamber to remain the same while increasing the area which the sandstone above the pyramid sits on, hence reducing the pressure on the pressure points. (as seen in the diagram above)

Firstly, Let the length of the side of the pyramid be x *m*, the length of the chamber to be y *m*, and the height of the pyramid to be p *m*.

$y = x(p-h)/p$.

The mass of the top part of the pyramid= $\frac{1}{3}[x(p-y)/p]^2(p-y)(2323)$

$= [2323x^2(p-y)^3]/3p^2$

The area that this weight is spread on $=(x(p-y)/p)^2 - y^2$.

For the pyramid to be able to support the weight,

$\{[2323x^2(p-y)^3]/3p^2\} \;/\; [(x(p-y)/p)^2 - y^2] < 5500000$.

For the maximum kings chamber size,

$\{[2323x^2(p-y)^3]/3p^2\} \;/\; [(x(p-y)/p)^2 - y^2] = 5500000$.

Using the formula that we found above, we were able to create a table of values to show the trend of how big a pyramid was and how big the king's chamber would be.

Height of pyramid / m	Length of base / m	Length of KC / m (1dp)

100	100	49.9m
200	200	99.6
300	300	149.2

From this table, we are able to see that the larger the size of the pyramid, generally the larger the size of the king's chamber in the pyramid.

RQ2 How does varying n, the number of sides of the pyramid affect the size of the king's chamber?

For this question, we decided to take a two step approach to solve it. We firstly tried to compare, on a two dimensional plane, the maximum size of square that can fit in a regular shape of a certain number of sides as we thought that this would help in finding the largest possible cubic king's chamber on a 3 dimensional plane.

To do this,

We Let the length of the side of the maximum fitting square be X.
Let the length of the side of the polygon be A.
Let the number of sides of the polygon be Y.

If Y = 3,
X = 0.464 A
If A =1000, X = 464
Square area = 215296

S = 1500
Area = 433012.701892

sqrt(500*500*500*1500) = 100*100*5*5sqrt(3)
= 5(6)*2(4) sqrt(3)
= 250000 sqrt(3)

Area of A/area of X = 250000 sqrt(3)/215296
= 2.01124359901

If Y = 4,
X = 1A
Both are squares thus both will overlap and so
Area of A/area of X = 1

If Y = 5,
X = 1.067395A
57.905920

X = r/sin(81)*sin(54)
r is the radius of the circle inscribing the pentagon.
Area of square = 2X^2 = 2(r/sin(81)*sin(54))^2
Area of pentagon = (r*r*cos(36)*sin(36))*5
Area of pentagon/ Area of square = (r*r*cos(36)*sin(36))*5/ (2(r/sin(81)*sin(54))^2)
= 1.77190691704

If Y = 6
X = 1.268A
Area of Hexagon = ((3/2)*sqrt(3))
Area of Square = 2(sin(60)/sin(75))^2
Area of Hexagon/ Area of Square = ((3/2)*sqrt(3))/ (2(sin(60)/sin(75))^2)
= 1.6160
If y = 7
1 = q0.64283
X = 1.555621A
Area of Heptagon = r*r*sin(180/7)*cos(180/7)*7
Area of Square = 2(r*sin(450/7)/sin(495/7))^2

Area of Heptagon/ Area of Square = (r*r*sin(180/7)*cos(180/7)*7) / (2(r*sin(450/7)/sin(495/7))^2)
= 1.50164842966

If y = 8
1 = q0.5412
X = 1.847745A
Area of Octagon = (r*r*cos(45/2)*sin(45/2))*8
Area of Square = 2(r*sin(135/2)/sin(135/2))^2

Area of Octagon/ Area of Square

= (r*r*cos(45/2)*sin(45/2))*8/ (2(r*sin(135/2)/sin(135/2))^2)
= 1.41421356237

y	3	4	5	6	7	8
X/A =	0.464	1	1.067395	1.268	1.555621	1.847745
Area of A/area of X	2.01124359901	1	1.77190691704	1.6160	1.50164842966	1.41421356237

As seen from the results , with a larger Y X/A get bigger and bigger infinitely. But for Area of A over Area of X when y = 4 it has the smallest value with Y=3 having the largest value.

Next, we had to construct this on the 3d plane, but because we didn't want to create many more equations relating to the individual kings chambers relating to their respective pyramids as that would be very time consuming, we decided to experiment with 4 different pyramids of different numbers of sides, of roughly equal size, and see how big the largest king's chamber that could fit in them could be.

As there was no real way of making all the pyramids the exact same size as in doing so, some would be wider than others or some would be taller than others, we assumed:

Radius of all pyramids = 100

Height of all pyramids = 100

For a Triangle pyramid of radius 100 height 100
X = length of cube
S = side of pyramid
x=100*sqrt(3)
x=s/(1+(2/sqrt(3))+sqrt(3/2))
≈0.2959065405s

Cube length = 51.25(4s.f.)
Length of sides of pyramid at the height of the maximum kings chamber

= 2X100cos(30) X48.75/100
=84.44(4s.f.)

Area of cross section of pyramid at height of kings chamber (By heron's formula)
S = 84.44x3/2
= 126.7 (4s.f.)
Area = sqrt(s(s-84.44)cube)
=3087m2

Area remaining = 3087- 51.25 X51.25
= 460.4 m2 (4s.f.)

Volume of sandstone above kings chamber = 3087 x 48.78 /3 = 50160 m3 (4sf)
Weight of sandstone above kings chamber = 2323 X 50160 = 116500000 (4sf)

Pressure on the pressure points = 253326 kg/m2
This is less than the structural strength of sandstone hence no shrinking is required

For a Square pyramid of radius 100 height 100
x=20*sqrt(50)
p=100
Using formula in RQ1:
Max cube length = 58.5m

Pentagon side length = 54.2497575874
cube length = 2h = 57.905920
Square area of maximum cube = 3353.09557105
Pentagon area at the height of the largest cube = 5063.42727
Difference = 1710.331698
Volume of top of pentagonal pyramid = 71046.77083
Weight of top of pentagonal pyramid = 2323X71046.77083
= 165041648.6

Amount of pressure on pressure point = 96496 (0dp) kg/m2
< 5500000
Hence no shrinking is needed as the pyramid is able to hold itself together.

cone of radius 100 height 100

By using the formula of the largest cube inscribed in a regular cone,
a = h*r√2/(h+√2*r)

Where a is the length of the cube, h is the height of the cone and r is the radius of the cone.

Length of side of cube = (100 X 100 sqrt(2))/(100 + 100sqrt(2))
=58.58m (4s.f.)

Volume of sandstone above kings chamber = pi 41.5X41.5X41.5/3
= 74800 m3(3s.f.)
Weight above kings chamber = 2323 X 74800
= 173000000kg (3s.f.)
Area of pressure point = pi 41.52 - 58.52
= 1990 m2
Pressure on pressure point = 173000000/1990
= 86900 kg/m2(3s.f.)
< 5500000 kg/m2
Hence no shrinking is needed.

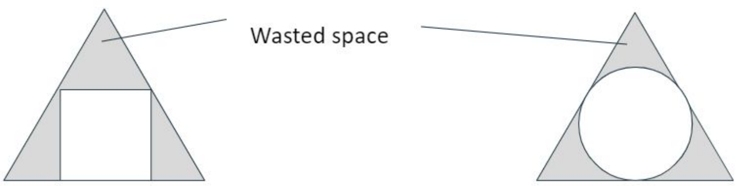
No of sides of pyramid	3	4	5	Infinity
Max length of kings chamber (m) (4s.f.)	51.25	58.85	57.91	58.58

As seen in the diagram above, we could not find a conclusive trend for our calculations. This may be partly because all the pyramids were not exactly of the same size, as there was no real way of making the pyramids the same size in terms of volume without messing up their heights or widths.

RQ3 Given the size, angle and number of sides of a pyramid, is there a way to calculate the largest possible kings chamber that can fit inside it given it can be of any shape?

When attempting this question, we realised that this question is actually extremely open ended, as there are many shapes to choose from a sphere, a cube, a cuboid ect.

However, through some experimentation, we realised such shapes often resulted in large amounts of wasted space that not only did not contribute to the size of the king's chamber, but also increased the weight that the pyramid had to support. (seen below)



Hence to maximise the available amount of space, we decided to make the king's chamber a pyramid, or basically hollow out the pyramid and make the 4 walls of the pyramid 4 separate slanted congruent pyramids (seen below)

Using the above construction, and since the equation to find the total volume of a slanted pyramid is exactly the same as that of a straight pyramid of equal height,

Hence we could find:
Pressure at the bottom of a fully filled pyramid= 2323 length width h/3 /length width
= 2323/3 h kg/m^2

As seen, the calculation for pressure at the bottom of the fully filled pyramid is not dependent on the length or thickness of the pyramid, hence theoretically the walls can be indefinitely thin. This also means that there will be a limit to how high this pyramid can go.

Upper bound for height of pyramid = 3 X 5500000 / 2323
= 7103m (4s.f.)

As seen, the upper bound for the maximum height of such a pyramid is 7103m.

4. Conclusion

As seen in the findings above, we have found out that without including any other external factors such as wind or human interference, the way the ancient Egyptians constructed the pyramids, a square based pyramid with a cuboid king’s chamber, was inefficient in maximising the total king's chamber size, and that In reality, If the pyramid were to still conform to the beliefs that the pyramid must have 4 faces, to maximise the size of the kings chamber one must construct a pyramid shaped king's chamber in the pyramid.

Of course our project and study does have some limitations, such as not factoring in external factors such as wind or human interference, or other internal factors such as the gaps between blocks in the pyramid or that not all sandstone is of one weight or structural strength. Hence, if there were to be a project extension, we would likely attempt to factor in such factors to provide a way to build pyramids that is much more applicable to the real world.

(1480 words excluding calculations, references, diagrams, tables ect.)