UNTANGLED

DISCOVERING PLANARITY

WRITTEN REPORT

Group 8-16
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1. Introduction

Description of Ideas

Untangled is a game that involves edges and vertices. The goal of the game is to move the vertices such that no edges intersect. Fig. 1 is an example. Notice that edges AC and BD in the graph intersect each other. Thus, moving vertex C into triangle ABD would “untangle” the graph.

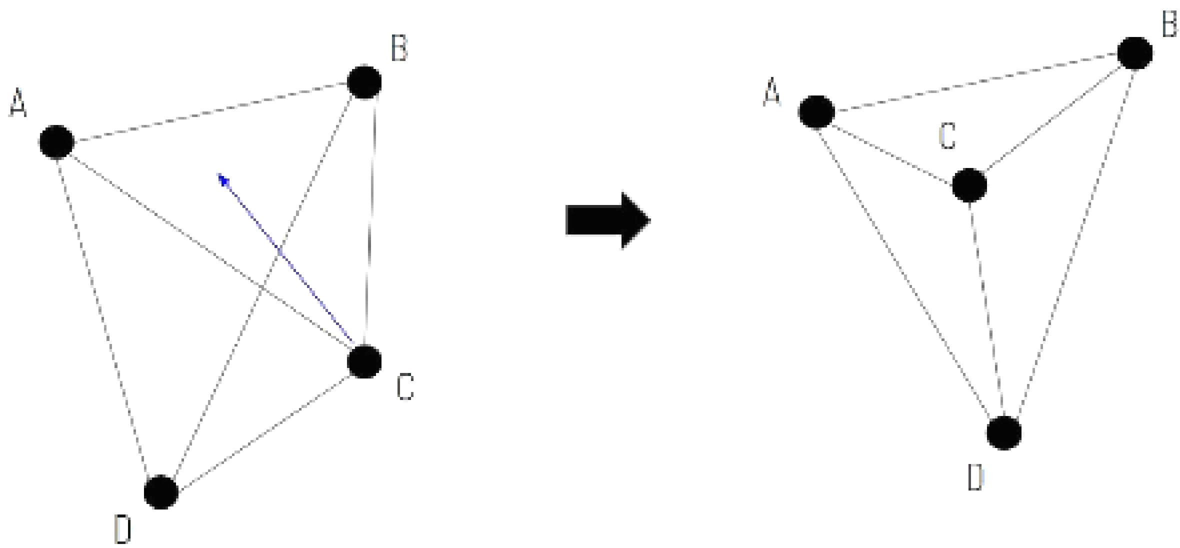


Fig. 1

Rationale

While looking through math games, the game Untangled caught our eyes. Unlike other games, Untangled is a one-player game that relies on the basis of untangling a planar graph. This appealed to us greatly and we found the idea behind this game especially interesting. From first glance, we could tell this game would involve graph theory, something that we have never been exposed to before. This would be a perfect opportunity for us to learn about and be exposed to a new field of mathematics that was previously alien to us!

Objectives and Significance

The objective of our project is to find strategies for both the original version of Untangled, and our own unique twist on the game (see page 15). We would then verify these strategies mathematically to improve accuracy. Our project is significant in a sense that we not only provide a fresh and intriguing perspective to an otherwise dull and repetitive game, but also provide original, verified strategies for our version of the game that can be used for future extensions of Untangled.

Research Questions

Our Research Questions are as follows:
Research Question 1: What is the strategy for Untangled?
Research Question 2: How do we solve the puzzle with minimal number of moves and verify it mathematically?
Research Question 3: What is the strategy for Untangled if certain vertices cannot be moved?

Scope of Study

Our scope of study throughout the project mainly focuses on the mathematical fields of Combinatorics and two-dimensional Graph Theory.

Terminology

Terms	**Definitions**
Graph	A graph is a **figure** made up of **vertices and edges**, drawn on a **flat plane**.
Subgraph	A subgraph is a graph all of whose **points and lines** are **contained** in a **larger graph**.
Planar graph	A planar graph is a graph that is **embedded in a plane** such that **no edges intersect**.
K_n	$\mathbf{K_n}$ denotes a graph where **n distinct vertices** are connected to **all other vertices with edges**.
$K_{n,m}$	$\mathbf{K_{n,m}}$ denotes a graph where **n distinct vertices** are connected to **m other vertices with edges**.

Hamiltonian cycle	A Hamiltonian cycle is a **closed loop** on a graph where **every vertex** is visited **exactly once**.
Move	A move refers to **moving a vertex** such that the **intersections change**. If moving a vertex does not change the intersections in any way, it does not constitute as a move.

Fig. 2 below shows a K_5 non-planar graph and Fig. 3 shows a $K_{3,3}$ non-planar graph.

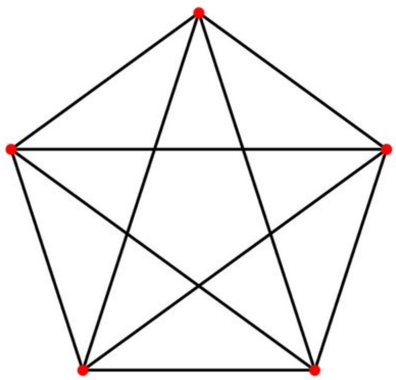


Fig. 2

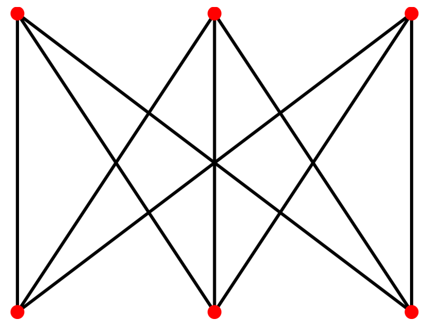


Fig. 3

Fig. 4 on the right shows an example of a Hamiltonian cycle. As shown, a red line starts from vertex A, passes through all vertices exactly once and goes back to vertex A.

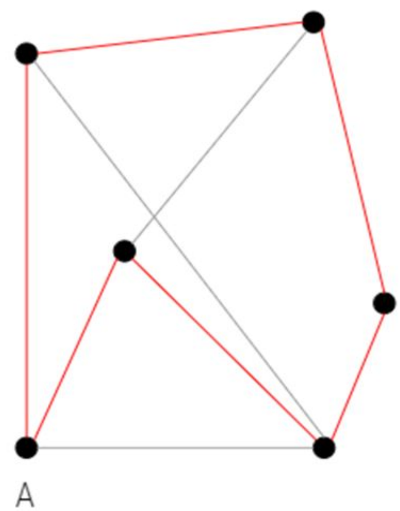


Fig. 4

2. Literature Review

Resources

In order to complete our project, we read up extensively on graph theory so as to better understand the theories that we might need to utilise in order to formulate comprehensive strategies for Untangled. Below are some of the resources that we found extremely helpful:

John Tantalon’s (2005) original version of the game (*Planarity*) gave us a basic understanding of the game. The version of the game allowed us to find patterns in solving methods, and later allowed us to test our theories and conjectures that we had in the process of formulating the strategy of Untangled.

Jeff Trim (2018)’s teaching resource, *Planarity*, was highly comprehensive in providing us with necessary concepts and insights that we would later find useful in obtaining the strategy of Untangled. We learned from it the meanings of planar graphs, as well as non-planar graphs such as K_5 and $K_{3,3}$ non-planar graphs while studying the Kuratowski's theorem. The resource also contained the definition of a Hamiltonian cycle, which would be integral in allowing us to formulate a strategy for the research questions. Lastly, we adapted the strategy of the teaching resource as the solution to Research Question 1. In conclusion, this teaching resource was highly critical in allowing us to learn about new terms and ways to approach the game of Untangled.

Peter Wentworth’s (2010) blog provided us with key insights on the strategy of Untangled. We found that this blog further reinforced the strategy mentioned in Jeff Trim’s resource and were able to gain new insights into planar graphs, as well as the appropriate usage of edges, vertices, and corners.

The Kuratowski’s Theorem

We will now introduce Kuratowski's theorem, which equipped us with the ability to distinguish between planar and non-planar graphs throughout the project. This is because non-planar graphs, like **K_5 or $K_{3,3}$** cannot be “untangled” by the rules of our project.
Kuratowski's Theorem states that **a finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or of $K_{3,3}$.**

3. Study and Methodology

Strategies and Methods to develop our project

As previously mentioned, we did thorough research in order to properly understand the game and the graph theory behind it. We went online to read up on the game and found several research papers that we found useful, found in our literature review. Apart from that, we also playtested the game during our free time. We downloaded a mobile app of the game Untangled which had more than 120 levels of increasing difficulty. By testing easier levels and moving onto harder ones, we managed to observe several patterns in the game that aided us in finding our solutions. Upon finding the solution, we incorporated previous findings so as to mathematically prove the strategies for Untangled, and to ensure that our solution was comprehensive and accurate.

4. Results and Findings

We will now discuss our findings for the three research questions to solve these questions. We have also come up with proposed solutions to answer the research questions.

Research Question 1: What is the strategy for Untangled?

Here is our proposed strategy for Research Question 1. Our strategy was adapted from a solution that we found in one of our resources.

1. Find a Hamiltonian cycle containing all vertices.
2. Re-construct the graph with this Hamiltonian cycle as an outer polygon with all other edges inside.
3. Select any inside edge and assign this to Set P.
4. Assign all edges that cross the first edge to set Q.
5. Assign all edges that intersect with edges belonging to Set Q to Set P, etc.
6. Redraw the graph with edges in set P drawn inside the cycle and edges in set Q drawn outside (or vice versa).
7. Shift points until the graph becomes planar.

We will now break down the strategy with an example, Fig. 5.

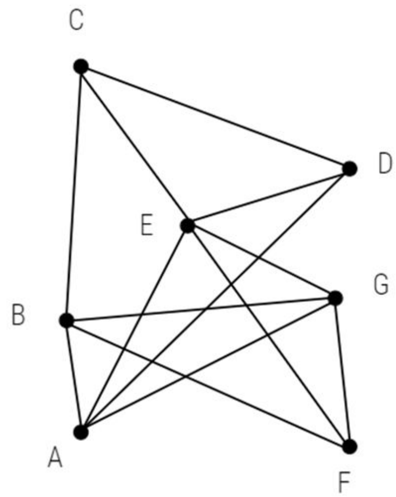


Fig. 5

Step 1

We first find a Hamiltonian cycle that contains all vertices of the graph. Fig 5.1 shows the Hamiltonian cycle for the graph above, highlighted in red.

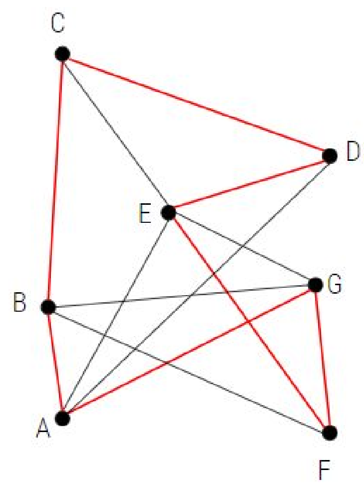


Fig. 5.1

Step 2

By moving certain vertices, we re-construct the graph such that the Hamiltonian cycle is an outer polygon encompassing all edges. As shown by Fig. 5.2, after moving vertices E and G, we achieve the above condition.

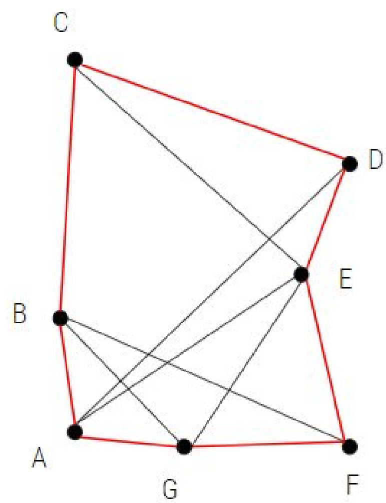


Fig. 5.2

Step 3

We select any edge that is inside the graph and assign it to a set P. As shown in Fig. 5.3 below, edge AD has been assigned to set P, and note that edges in Set P will be marked out with blue lines.

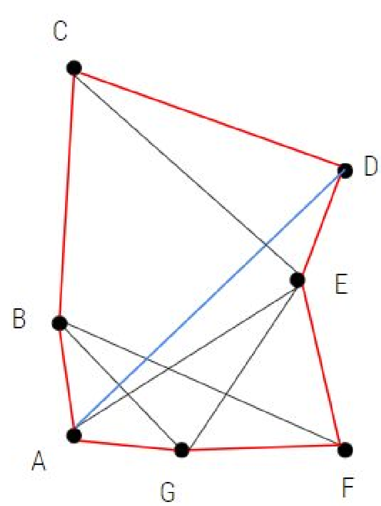


Fig. 5.3

Step 4

We assign all edges crossing the edge in Set P to Set Q. In Fig. 5.4 below, edges CE, BF and BG intersect AD, and have thus been assigned to Set Q, marked out with purple lines.

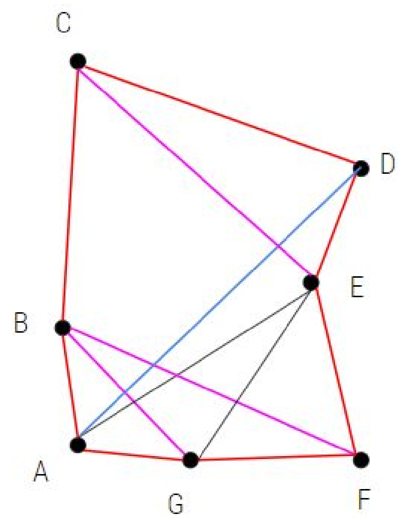


Fig. 5.4

Step 5

We then assign all edges that intersect any edge in Set Q to Set P, repeating the process until all edges have been assigned to either set. As shown in Fig. 5.5. Edges AE and GE have been assigned to Set Q here.

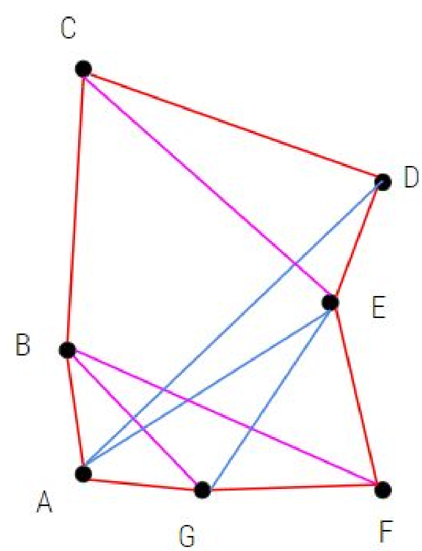


Fig. 5.5

Step 6
Finally, we redraw the graph with edges in Set P drawn inside the Hamiltonian polygon and edges in Set Q drawn outside of it, or vice versa. In Fig 5.6, Set P has been drawn outside the Hamiltonian polygon.

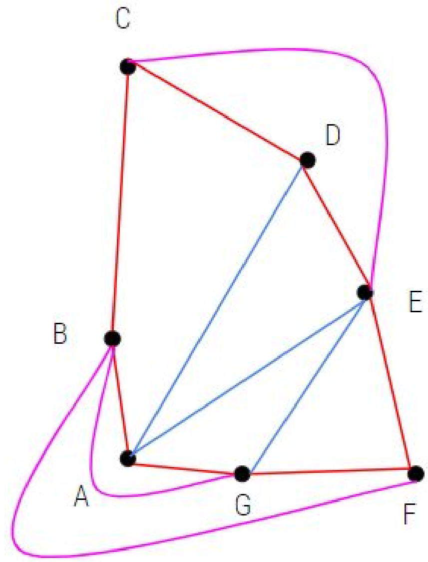


Fig. 5.6

Step 7
It has been proven that there will always be a way to represent graphs with curved edges with straight edges. Redrawing the previous graph, we get Fig. 5.7.

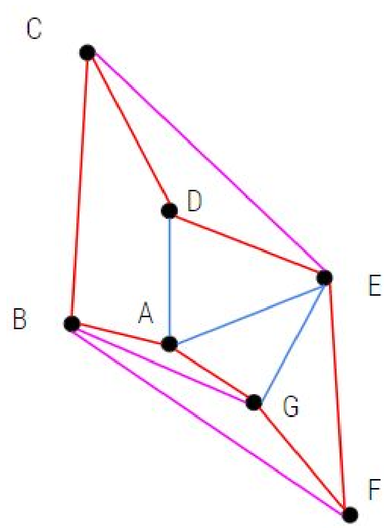
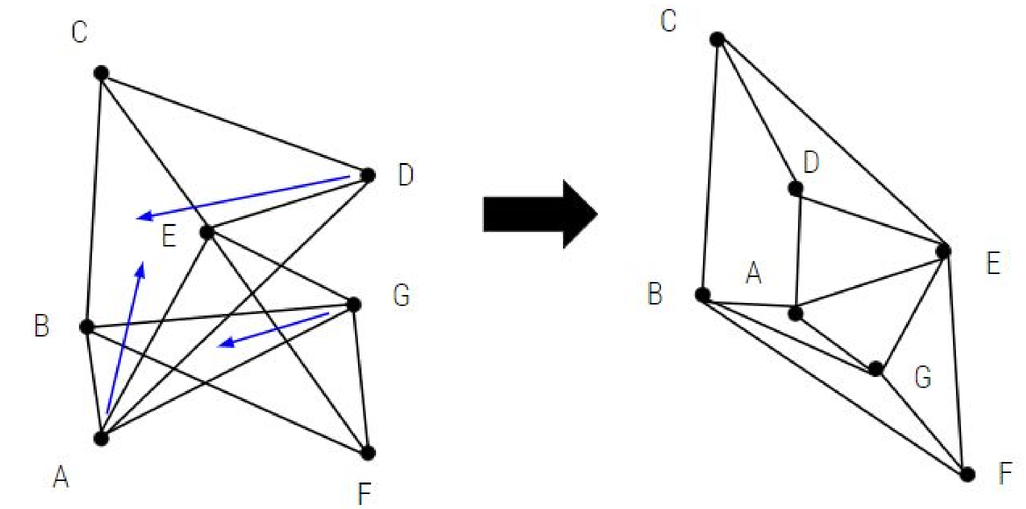


Fig. 5.7

From here, simply move the vertices such that the graph resembles the newly constructed, planar one!



Since no edges in a single set will intersect with one another, our strategy will guarantee that the newly formed graph will in fact be planar! [1]

Research Question 2: How do we solve the puzzle with minimal number of moves and verify it mathematically?

Here is our solution for Research Question 2 , which we have built upon that of Research Question 1. As with Research Question 1, we will illustrate this solution with an example!

1. In the graph, search for the vertex that is connected to most number of intersections, then the one connected to least number of vertices (if applicable).
2. Move the vertex such that most number of intersections are reduced.
3. Repeat Steps 1-2 until the graph is solved.

We will use the same graph from Research Question 1, Fig. 5.

Step 1

As shown in Fig. 6.1, we first search for the vertex connected to the most number of intersections. In the event that there are more than one such vertex, we choose the vertex connected to the least number of vertices. Here, vertex A is the vertex we are looking for.

[1] Note that in the rare occurrence (e.g K_5) that vertices in a single set intersect, that graph is non-planar and cannot be solved.

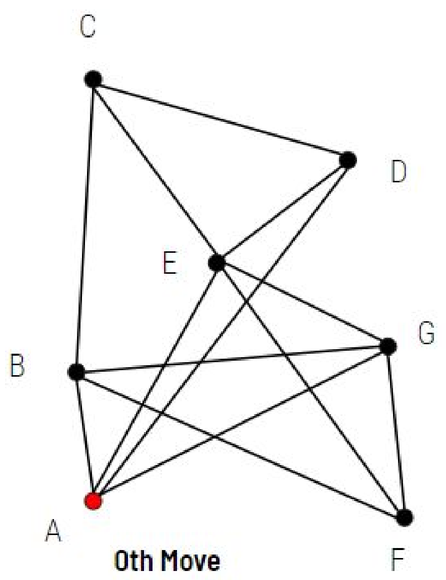


Fig. 6.1

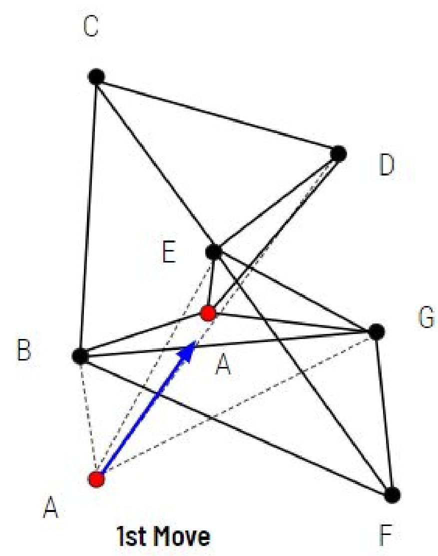


Fig 6.2

Step 2
From here, move the chosen vertex such that the number of edges that intersect is reduced to the minimum possible number. In Fig 6.2 above, moving vertex A reduces the original 8 intersections extending from it to 3, the minimum possible number of vertices that intersect.

Step 3
Finally, repeat Steps 1 and 2 until the graph is solved. Here, A and E are both connected to 3 intersections. However, since each vertex only needs to be moved once to solve a graph, moving A would be redundant, as it would not reduce the intersections connected to A to less than 3. Thus, Fig. 6.3 and Fig. 6.4 show Steps 1 and 2 being repeated on vertex E before the graph is solved. Hence, a minimum of 2 moves are needed to solve the graph.

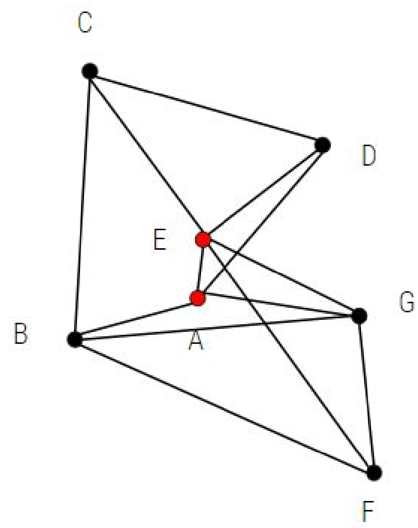


Fig. 6.3

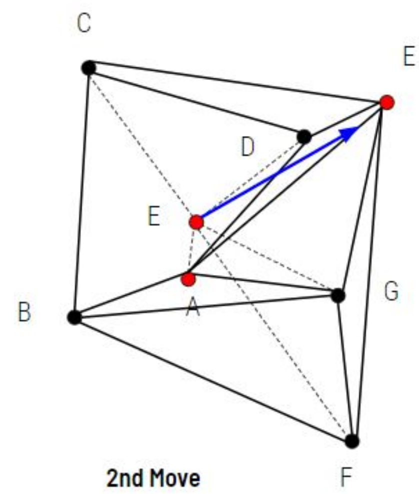


Fig. 6.4

Mathematical Reasoning

By solving a graph, we are reducing the number of intersections in the graph to 0. Since we aim to reduce the most number of intersections each move, thus, finding the vertex connected to the most intersections would allow us to remove the most number of intersections.

When there are more than one vertex connected to the most number of intersections, however, we choose the one connected to the least vertices, as it would produce the least number of new intersections when being moved.[2]

Thus, Steps 1 and 2 would logically ensure that every move reduces the most number of intersections, eventually solving the graph with minimal number of moves.

[2] The number of intersections reduced is determined by the net difference of intersections removed and formed.

Research Question 3: What is the strategy for Untangled if certain vertices cannot be moved?

For Research Question 3, we stated that solving a graph of the new version of Untangled would involve the same procedure as that of a normal Untangled graph. The restriction would only affect us finding the Hamiltonian cycle. We will prove this in subsequent pages.

Mathematical Reasoning

We will first discuss the cases of edges in a Hamiltonian polygon.

1. When an edge is formed by two free vertices:
   When assigning it to Set P, the edge can move, with two free vertices
   When assigning it to Set Q, the edge does not need to be moved.

2. When an edge is formed by one free vertex and one restricted vertex:
   When assigning it to Set P, the edge can move by pivoting the free vertex around the restricted vertex.
   When assigning it to Set Q, the edge does not need to be moved.

3. When an edge is formed by two restricted vertices:
   Since the edge cannot be moved, it cannot be assigned to Set P.
   Thus, we can simply assign the edge to Set Q, where it does not need to be moved.

We have thus concluded that no matter the composition of an edge, it can always be accounted for in a Hamiltonian cycle. You might have noticed, however, that if we assign an edge with two restricted points to Set Q and another such edge intersects it, that edge will have to be assigned to Set P, which is not possible. This will bring us to cases with no possible solution.

Cases where no solution is possible

We could find 2 general cases that would never have a solution.

1. When the restrictions make it such that **the number of movable vertices is less than the minimal number of moves required to solve the original graph.**
   The minimal number of moves can be derived through the method used in Research Question 2 (see page 12). Since we are using the same graph in this question as Research Question 2, we have previously found out that at least 2 moves are required to solve it. Fig. 7.1 shows one such example. Theoretically, each vertex only needs to be moved a maximum of one time to solve a graph. With that in mind, since the number of movable vertices is less than the minimal number of moves to solve the original graph, we will have

to solve the graph with a number of moves less than the minimum number of moves needed to solve it, which is a contradiction, and hence impossible.

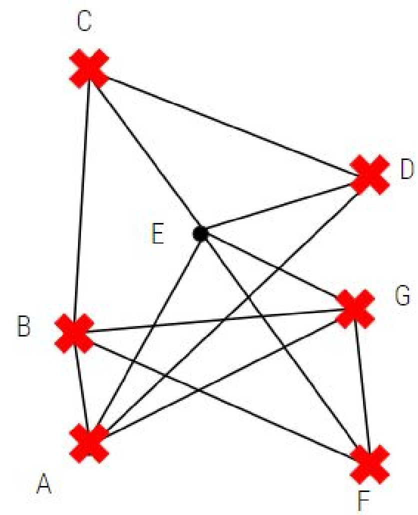


Fig. 7.1

2. When **4 points are restricted** such that they form a **kite**.
   As discussed previously, these two edges formed by the vertices intersect each other and cannot be untangled. Fig. 7.2 shows a general example of this case.

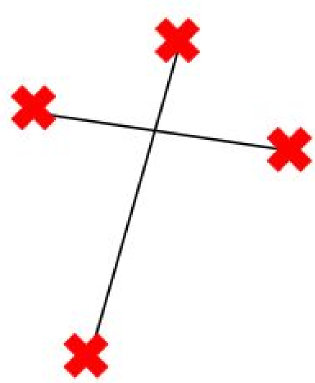


Fig. 7.2

5. Conclusion

Reflection and Learning Points

To us, this math project was not just enriching for our minds, but also taught us many important life skills that would certainly be of use in the future:

1. Time management. This aspect of a project might seem small and insignificant, but it can really make or break a project. Though our project was relatively smooth, there were definitely times where the lack of time management cost our productivity in making progress. Especially in this time of crisis, we learned that with a good schedule and plan, half the battle is already won.
2. Communication. Good communication is key to a team's success. We could especially see this in the circuit breaker period, where we could not meet up physically to work on our project. Because of this we had to communicate efficiently and meet up online regularly to successfully get our work done.
3. Procrastination does us no good. Procrastination only pushes the workload to future days.
4. A seemingly small problem could evolve into something critical in the future. We learned that we should always solve the problem at hand, and never leave something untouched.

Possible Extensions

1. Our solution, though accurate, working and feasible, is not practical for complicated graphs with tens of hundreds of vertices. Here, computationally coding our solution into a program can allow us to solve this problem far more efficiently. (RQ 2)
2. Apart from our mathematical proofs for RQ 1, 2 and 3, we can also computationally verify them for improved accuracy.
3. Untangled is a fluid, flexible game. We can certainly explore more variations of the game. (e.g. 3-Dimensional Untangled)

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