Cannibals and Couples

Written Report

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Abstract

Suppose that there are 3 couples that want to cross the river. The husbands are very jealous, and refuse to let their wives be with other men unless the husband himself is there too. Unfortunately, there is a boat that only holds 2 people at a time. How can all 3 couples cross the river in the minimum number of moves? This paper aims to investigate the multiple approaches to this problem, and find a standardised strategy for the original problem as well as variations. We found that recurrence solving worked best, especially since it could be tweaked to fit many variations, systematically sending couples to the other bank. Our strategy is easy to understand, with minimum knowledge of algebra required.

Contents

1 Introduction

The River Crossing Problem, specifically the Jealous Husbands Problem, has received much attention in the past. The fascinating aspect of the problem is how it can be solved with many different techniques and strategies. This project aims to explore various strategies and solve some variations as well.

1.1 Objectives

1. To find a strategy for the smallest number of moves to cross the river in the original problem
2. To find out if the strategy still applies when there are different number of people
3. To explore more variations of the game

1.2 Research Questions

1. What is the strategy of the original river crossing problem, with minimum number of moves?
2. How do we generalise our result when there are varied number of people crossing the river?
3. How will the strategy change when the other parameters are varied?

1.3 Literature Review

The Jealous Husbands problem first appeared in Alcuin's collection *Problems to Sharpen the Youth* in around 800 AD. Since the early 1960s, these puzzles have been used as illustrations of artificial intelligence methods (Efimova, 2018). Many different methods are used to solve the problem: Schwarz (1992) uses adjacency matrix of the graph, Bellman applies dynamic programming, Fraley, et al. (1966) place the graph on the coordinate plane, etc.

Wallingford (2014) states that the Jealous Husbands problem is a case of using instance simplification, which is often a form of decrease-and-conquer, where one has to simplify the problem into an easier, solvable state. Pressman and Singmaster (1989) recognised that the solution should be split into stages, which can be repeated based on the value of n. This method of recurrence solving is used when each stage is repeated, simplifying the problem slowly. The 2 latter articles inspired our solving, giving us a comprehensive strategy to solve both the original problem and the variations.

1.4 Notation and Terminology

n:	Number of Couples
C (n):	Total Number of Moves
H:	Husband
W:	Wife
Original river crossing problem:	3 couples crossing in 2-person boat
Left bank:	River bank that couples start on
Right bank:	River bank that couples end on
Tree:	Diagram showing possible moves via nodes

2 Methodology

We first played the game, experimenting with various strategies, then did a thorough literature review, to evaluate the best strategies and their shortcomings. The literature review included looking at past articles as well as exploring possible methods of solving. We decided on using a methodical decrease-and-conquer strategy, then solved the original problem. Finally, we solved the variations, where *n* was undefined and where an island was added, using the same technique.

3 Results

This is our findings for the original problem as well as variations.

3.1 Findings for Original River Crossing Problem

3.1.1 Using Vectors

Vector addition and subtraction is one way to solve the jealous husbands problem. The method used is by using a vector, with (*a, b, c*): a being the number of wives on the left bank; b being the number of husbands on the left bank, and c being 1 if the boat is on the left bank while 0 if the boat is on the right bank. The initialisation is (3, 3, 1) as initially, there are 3 wives on the left bank, 3 husbands on the left bank, and 1 boat on the left bank too. Since the vector representing the right bank will be (3, 3, 1) minus (*a, b, c)*, there is no need to show that. The explanation for the vector is shown in Figure 1.1.

(a,   b,   c)

a: Number of wives on wrong side

b: Number of husbands on wrong side

c: Position of boat
- 0 = boat is on right side
- 1 = boat is on wrong side

Figure 1.1

Actions are represented using vector subtraction/addition to manipulate the state vector. For instance, if a lone wife crossed the river, the vector (1,0,1) would be subtracted from the state (3,3,1) to yield (2,3,0). The state would reflect that there are still two wives and three husbands on the wrong side, and that the boat is now on the opposite bank. A simple diagram can be constructed, as shown in Figure 1.2. From the diagram, we can easily see that there are 11 steps in solving the problem, discounting the starting orientation. The problem is finished when we see the transformation of (3,3,1) to (0,0,0).

Using vectors cannot show multiple solutions, however. Thus, we searched for a better way to present the solutions.

Figure 1.2

3.1.2 Using Trees

To fully solve the problem, a simple tree is formed with the initial state as the root. The five possible actions to move one wife across, move two wives across, move one husband across, move two husbands across and move one couple across are then *subtracted* from the initial state, with the result forming children nodes of the root. Any node that has a wife unaccompanied by her husband on either bank is in an invalid state, and is therefore removed from further consideration. For each of the remaining nodes, children nodes are generated by *adding* each of the possible action vectors. The algorithm continues alternating subtraction and addition for each level of the tree until a node with no more couples on the left bank is generated. The complete solution is shown in Figure 2. The red lines in Figure 2 represent invalid states and the green lines represent valid states. The arrow in the middle of each state represents where the boat is headed to *next*.

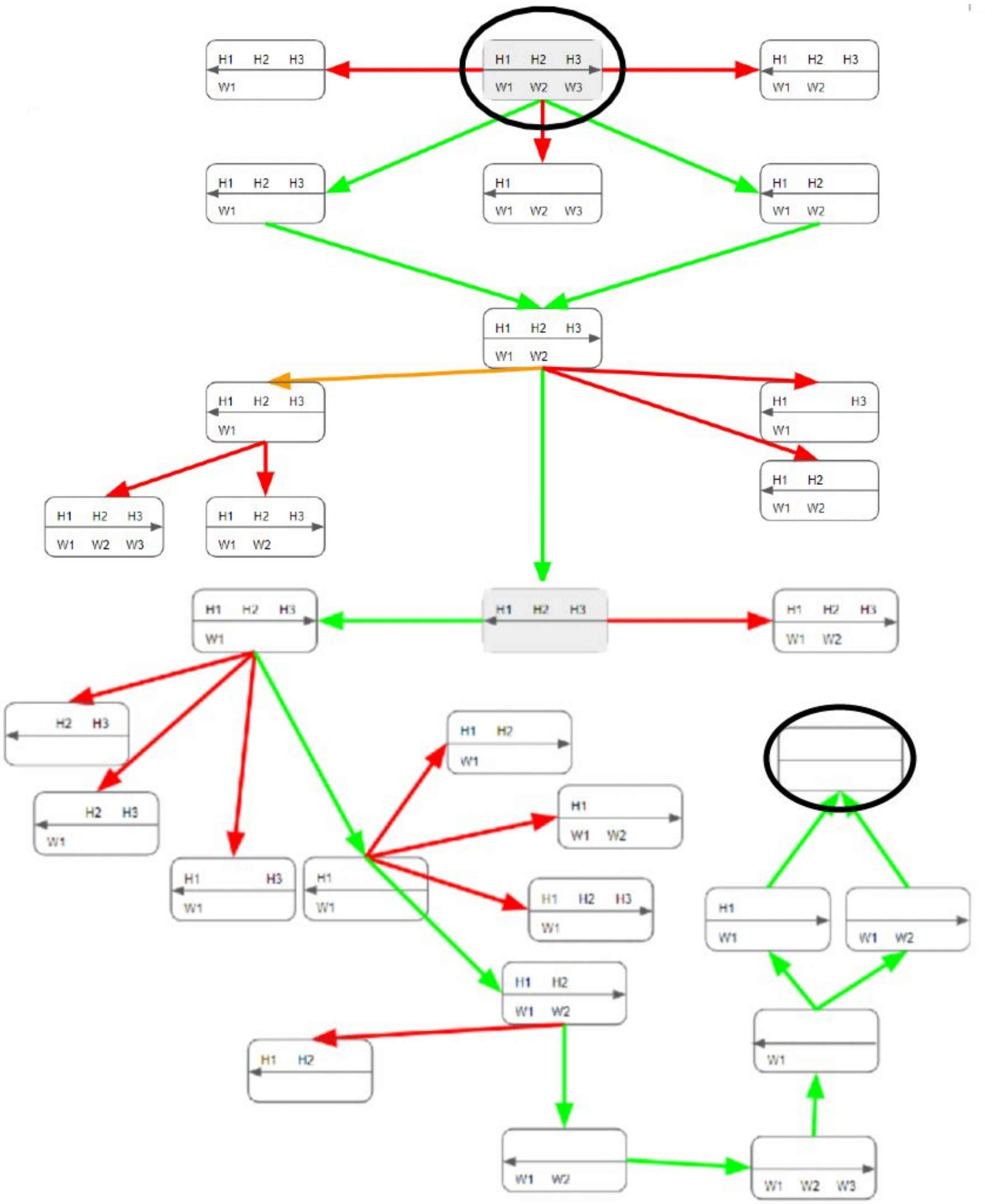

Figure 2

From Figure 2, we can easily see that it takes a minimum of 11 moves to solve the original problem. There are 4 methods, with slight variations at the beginning and the end.

3.2 Findings for Variations

Having solved the original problem, we now aim to observe any patterns and apply them to solve variations.

3.2.1 Varying Number of Couples

What happens when we change the number of couples, n? Is there still a strategy or formula to solve the problem?

We noticed that in the original river crossing problem, 1 couple was sent forward first (refer to Figure 3.1), simplifying the problem down into a 2-couple problem. We investigated further, and found the same trend in 4 and 5 couples. Each time, one couple was sent forward first, systematically simplifying the problem down from n couples into $n-1$ couples.

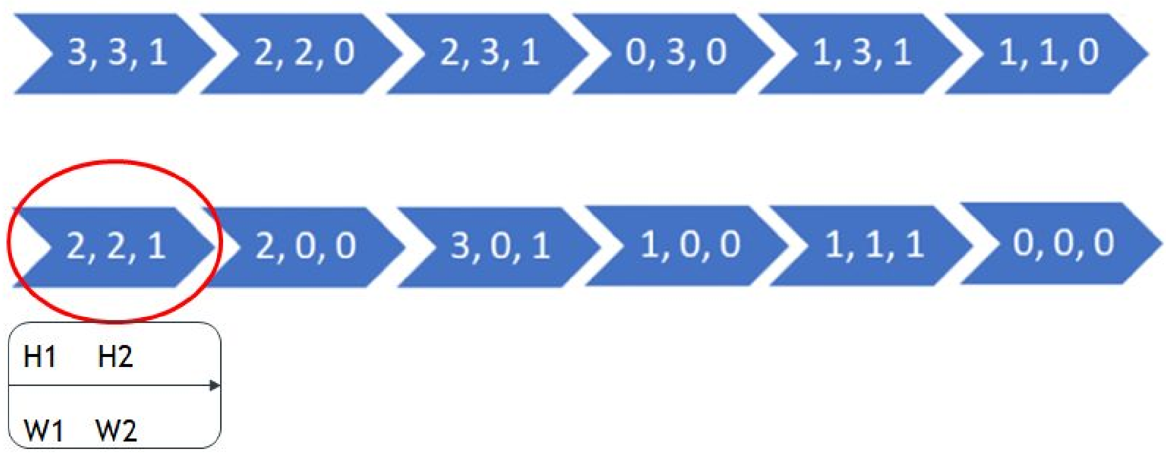

Figure 3.1

A couple was always sent to the right bank using the same 6 moves:

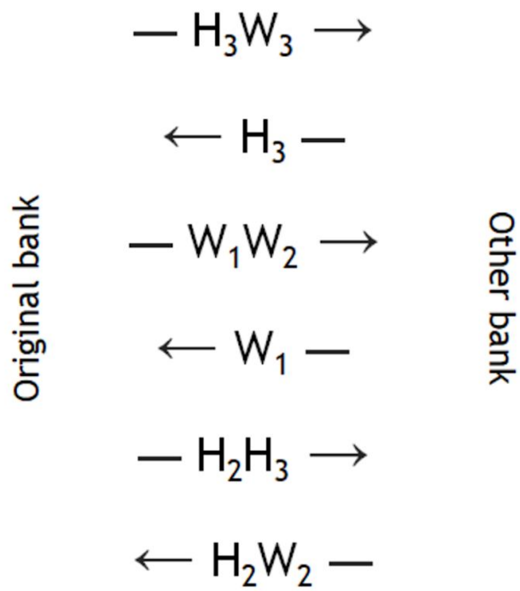

Figure 3.2

At a high level, we can describe the transformation from n couples to $n - 1$ couples as:

choose a couple, H_i and W_i to move

move H_i and W_i across

move H_i back

move both remaining wives across

move one of the other wives back

move H_i and another husband across

move the other couple back

The 6 step move shown in Figure 3.2 can be repeated until there are 2 couples left on the left bank, during which we will use this 5-step method to solve the problem.

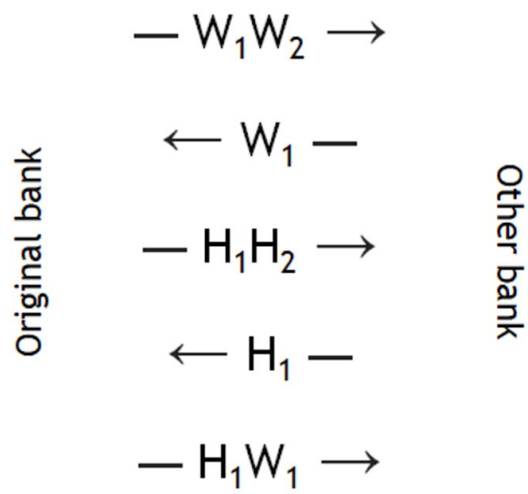

Figure 3.3

At a high level, we might describe solving the two-couple case as:

choose a couple, H_i and W_i, to move first:

move both wives across

move W_i back

move both husbands across

move H_i back

move the other couple

Incidentally, those 5 steps will not work when there are more than 2 couples left on the same shore, because the husband is required to leave his wife alone in step 3.

With the above information, we are able to obtain a formula. Since every 6 steps will move one couple to the opposite shore, we repeat the 6 moves until only 2 couples remain, during which we will use the 5 move sequence to move them across, thus solving the problem. Thus, $n - 2$ sets of the 6 move sequence leave 2 couples behind, which can then be solved by the 5 move sequence. Therefore, $6\,(n - 2) + 5$ equals the total number of moves used.

Our formula can be proven by observing the recurring pattern. We start with what we know: every 6 moves is able to change n couples to $n - 1$ couples on the left bank. Thus, we are able to draw this opening equation:

$$C\,(n) = 6 + C(n-1) \text{ for } n > 2$$

By repeating the 6-move sequence, the recurring relationship can be set up:

$$\begin{aligned} C(n) &= 6 + C(n-1) \\ &= 6 + [\,6 + C(n-2)\,] \\ &= 12 + [\,6 + C(n-3)\,] \\ &= 18 + [\,6 + C(n-4)\,] \\ &\;\;\cdots \end{aligned}$$

Once there are only 2 couples left on the original riverbank, we can see that the number of six-move-sequences we did is equivalent to n − 2. We can then replace them with n − 2.

$$\begin{aligned} C(n) &= 6\,(n - 2) + C\,(n - (n - 2)) \\ &= 6\,(n - 2) + C\,(2) \\ &= 6\,(n - 2) + 5 \end{aligned}$$

Since we have already concluded that the number of moves needed to move 2 couples is 5, we can prove that n number of couples can cross the river in $6\,(n - 2) + 5$ moves.

3.2.2 Changing Setting of Boat

If a woman in the boat at the shore, but not on the shore, counts as being by herself (i.e. Not affected by any men on the shore itself), then the puzzle can be solved in 9 trips.

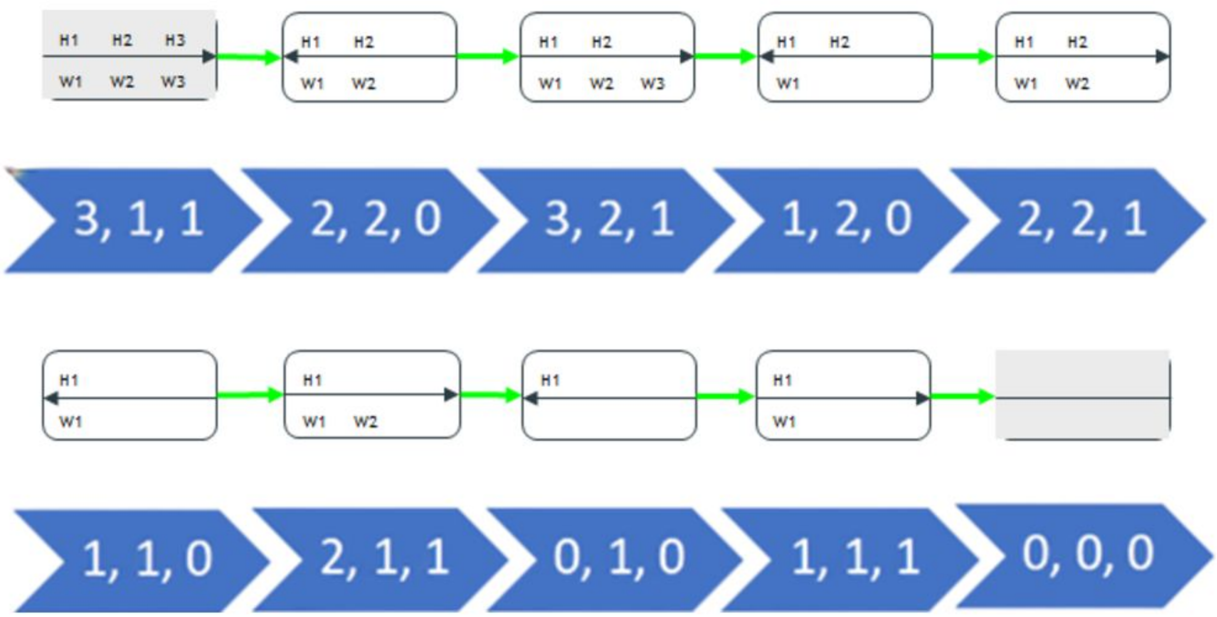

Figure 4.1

The boat acts as an extra compartment or “island”, thus the couples can cross in less steps.

3.2.3 Addition of Island

3.2.3.1 Prohibiting Bank-to-Bank Crossings

Crossing from the left bank straight away to the right bank, and vice versa, is not allowed in this variation.

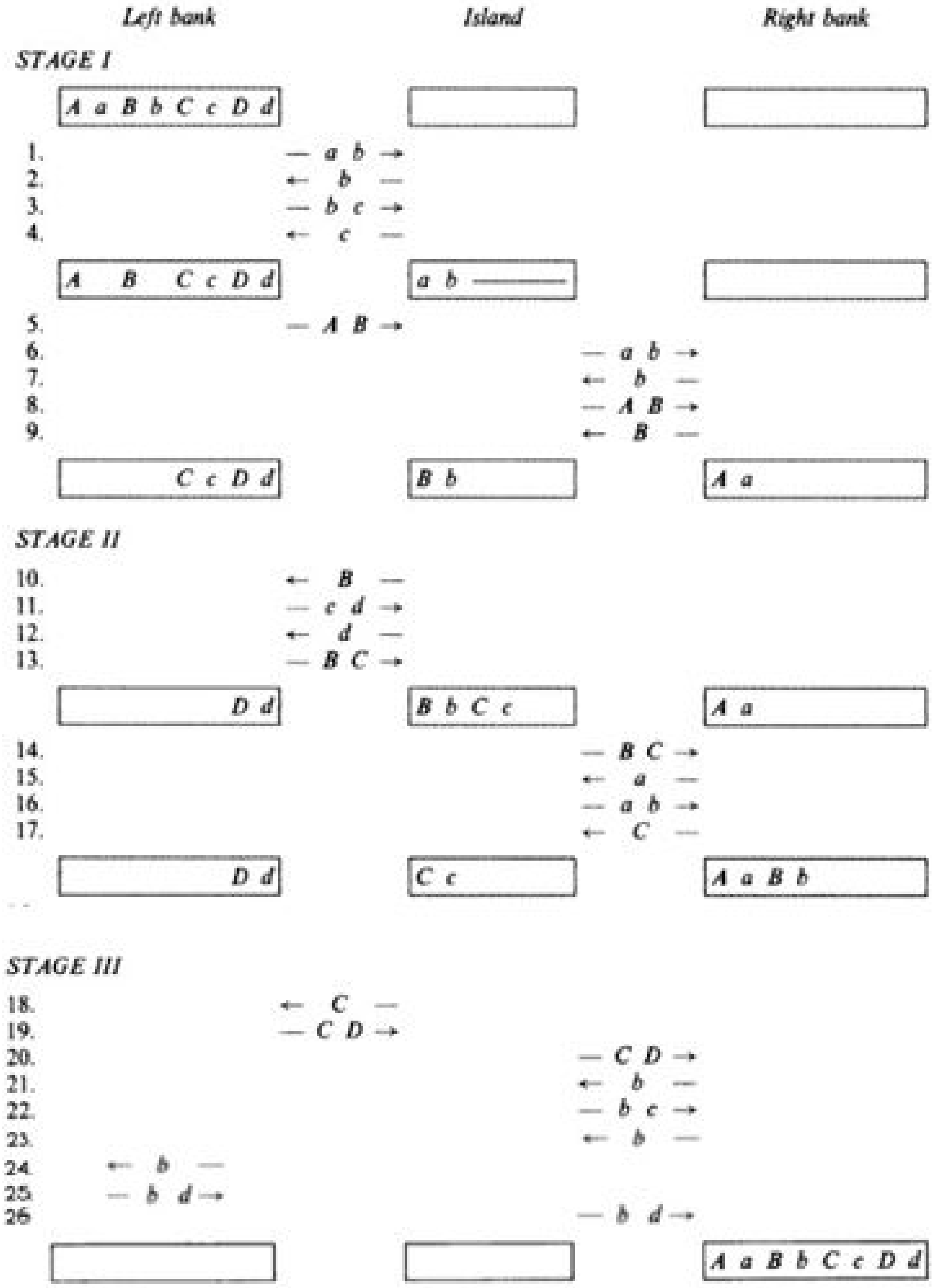

Figure 4.2

This is the overview of our solution.

What is interesting is the difference between the beginning and the end of the second stage (refer to Figure 4.3) is simply one couple moved from the left bank to the right bank. Thus, by applying the same decrease-and-conquer technique we used in varying n, we can repeat stage 2 multiple times, moving one couple to the right bank each time.

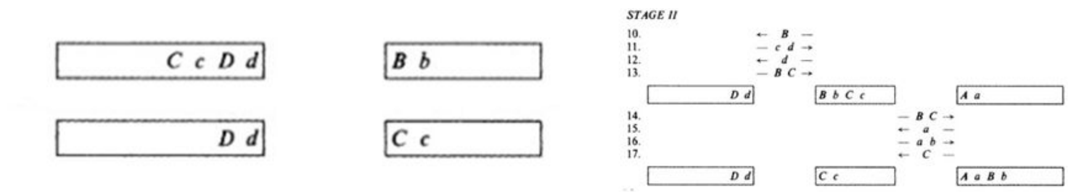

Figure 4.3

Stage 1 and Stage 3 combined move 3 couples across the river. Thus, by repeating Stage 2 ($n-3$) times, we are able to move ($n-3$) couples across the river.

With 8 moves in Stage 2 and 18 moves in Stage 1 and 3 combined, we can find the minimum number of moves if bank to bank crossings are prohibited, in terms of n.

However, this solution only works when n is larger than or equal to 4, because even if stage 2 isn't repeated, a minimum of 4 couples have to be sent.

$$C(n) = 8(n-3) + 18$$

$$= 8n - 6 \text{ for } n \geq 4$$

This is the proof for our formula, $8n-6$. Consider any method of ferrying $2n$ people across the river without bank-to-bank crossings. Each departure and return from the left bank results in at most one person being taken from the left bank to the island, since one person always has to drive the boat back. The exception is the last departure, where the last 2 people can go to the island.

Hence, there must be $2(2n-2)+1 = 4n-3$ crossings involved from the left bank to the island. 2 trips are required to send 1 person to the island everytime, and the last trip is when 2 people can cross, hence the -2 and $+1$.

Since the same holds true for moving n couples from the island to the right bank, the total number of moves required is $8n - 6$.

3.2.3.2 Permitting Bank-to-Bank Crossings

Crossing from the left bank straight away to the right bank, and vice versa, is allowed in this variation.

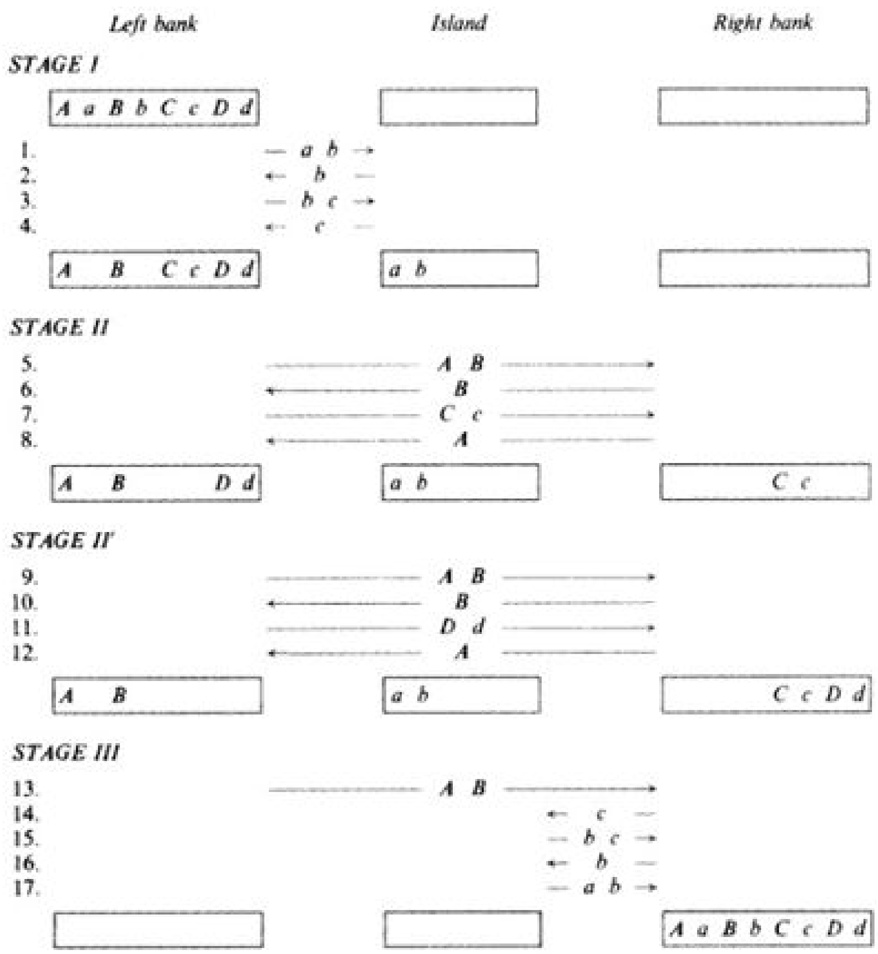

Figure 4.4

The same decrease-and-conquer technique is involved. We repeat Stage 2 $(n - 2)$ times to get $(n - 2)$ couples across.

$$C(n) = 4(n - 2) + 9$$

$$= 4n + 1 \text{ for } n \geq 3$$

Since all 3 Stages combined only move 3 couples across, this formula can be used as long as n is larger than or equal to 3, unlike in Variation 2a, where the formula only works when n is larger than 3.

4 Conclusion

Both the original problem and the variations make use of the same “decrease-and-conquer” strategy, systematically bringing one couple to the right bank first. In all the variations, the formula for number of moves makes use of this strategy.

5 Possible Extensions

One extension to the problem is known as the wolf, goat and broccoli problem. Basically, the problem is that a wolf, golt and a broccoli have to all get across a river. The wolf would eat the goat, and the goat would eat the broccoli. The problem is to get all three living things across the river without one eating the other. This problem is very similar to the Jealous Husbands problem but a little simpler due to the fact that there only has to be 3 different objects, compared to the 6 objects used in the cannibals and couples problem.

The Jealous Husbands Problem is akin to the Missionaries and Cannibals Problem. In the Missionaries and Cannibals Problem, there are 3 missionaries and 3 cannibals trying to cross the river. The number of cannibals cannot exceed the number of missionaries on any bank, or the cannibals will eat the missionaries. This plight is similar to the Jealous Husbands Problem since the number of wives cannot be larger than the number of husbands, or at least one wife would be without her husband. Thus, solving the Jealous Husbands Problem solves the Missionaries and Cannibals Problem.

6 Acknowledgements

A very special thanks to Mr Chen Zenghui for guiding us along this project, and for always patiently supporting us and giving us constructive criticism.

7 References

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