Magic of the Fibonacci Sequence

Project Report

Group 8-11

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Abstract

Given a game with k possible outcomes, each having an equal probability of occurring, this project seeks to find the probability that n rounds of the game are required to achieve m consecutive rounds with an identical outcome. The probability that n rounds are required was first expressed in terms of the sum of the probabilities that $n-j$ rounds were required for each integer value of j less than m. For this recurrence relationship, a characteristic equation was found and proven to be valid. On solving it, an explicit expression of the probability in terms of k, m and n was thus found. Generalisations of the solution and potential limitations to its applicability were then discussed.

Contents

Introduction

Rationale

Tom Stoppard's 1966 play *Rosencrantz and Guildenstern Are Dead* begins with the titular characters betting on coin tosses and tossing 92 consecutive heads, so many that Guildenstern exclaims, "A weaker man might be moved to re-examine his faith, if in nothing else at least in the law of probability", and perhaps even the Fibonacci sequence.

Such a bizarre situation falls under the study of simple games of chance with random outcomes, such as coin-tossing as well as dice-throwing. As mentioned in Grinstead and Snell (1997), the study of such games is an important field in probability theory, and one that deserves further research.

The Fibonacci sequence mentioned is also of interest here. The Fibonacci sequence, a recursive sequence where the first two terms are equal to 1 and each term thereafter is equal to the sum of the two preceding, "occupies a special position" in number theory and probability theory (Springer, 2011). Numerous applications of the Fibonacci sequence already exist in these fields, such as applications concerning continuing fractions, hyperbolic functions (Vajda, 2008), and even coin-tossing; Willard, Dobbs, and McQuillan (1986) proved that the probability that n tosses of a fair coin were required for two consecutive heads to appear was $\frac{u_{n-1}}{2^n}$, and the probability that n tosses were required for three consecutive heads was $\frac{u_{n-2}}{2^{n-1}}$, u_n being the n^{th} term in the Fibonacci sequence.

Given the significance of Willard, Dobbs, and McQuillan (1986), the Fibonacci sequence and the study of simple, random games such as coin-tossing, this project then seeks to generalise the results of Willard, Dobbs, and McQuillan (1986) to any number of consecutive heads -- even Guildenstern's 92, and to games where each round has more than two possible outcomes. Having done so, this project will then consider whether the probability found for these more generalised cases is still connected to the Fibonacci sequence, or otherwise some other mathematical sequence, model, or constant.

Objective

For a game where each round, and in particular the setup of each round, is identical; and where each round has one of k possible outcomes, these k outcomes being identical to the k outcomes of any other game; and where the outcome of any round of the game is independent of that of previous rounds, this project seeks to find a probability function for the probability that n rounds are required for the game to end, where the game ends immediately after the first occurrence of m consecutive rounds with an identical outcome.

Research Questions

1. What is the probability that n tosses of a fair two-sided coin are required for m consecutive heads to appear?

Also, for a game as described in the Objective, each round having k possible outcomes,

2. Where the game ends immediately after the first occurrence of two consecutive rounds with identical outcomes, what is the probability that n rounds are required for the game to end?
3. Where the game ends immediately after the first occurrence of m consecutive rounds with identical outcomes, what is the probability that n rounds are required for the game to end?

To answer these research questions, this project seeks to generalise the results of Willard, Dobbs, and McQuillan (1986), from the case of two consecutive heads to that of m consecutive heads, and from the case of a game where each round has two outcomes to one where each round has k outcomes. When these research questions are answered, it will emerge whether the probability found is still related to the Fibonacci sequence, or perhaps some other mathematical sequence, model, or constant.

Relevant Fields of Mathematics

1. Number theory, a branch of mathematics devoted primarily to the study of integers and integer-valued functions, not least the Fibonacci sequence
2. Probability theory, a branch of mathematics concerned with the likelihood that something has happened, will happen, or is true

Literature Review

There is literature that relates to our three Research Questions (RQs), but the scope of various sources tend to overlap, and are in any case different from the scope of our RQ, in particular RQ3. They nevertheless are instructive.

Willard, Dobbs, and McQuillan (1986) proved that the probability that n tosses of an ideal, fair coin are required until two consecutive heads appear was equal to $\frac{u_{n-1}}{2^n}$, and that the probability that n tosses of an ideal, fair coin are required until three consecutive heads appear was equal to $\frac{u_{n-2}}{2^{n-1}}$, where u_n is the n^{th} term in the Fibonacci sequence. These results were proven by considering two separate cases: the case where n tosses were required with the first toss resulting in a head, and the other where n tosses were required with the first toss resulting in a tail. It was then established that the probability of n tosses being required where the first toss resulted in a head could be expressed as a sum of the probabilities that $n-1$ and $n-2$ tosses were required where the first toss resulted in a tail, and vice versa. Willard, Dobbs, and McQuillan (1986) offer useful guidance, but their results are limited in utility, as their scope is limited to the case of up to 3 consecutive coin-tosses resulting in the same outcome. This project seeks to expand and generalise these results to the case of n consecutive coin-tosses to answer the first research question.

Griffiths (2011) used hyperbolic functions to prove that m , the number of pairs of consecutive heads (two consecutive heads) in n tosses of the coin, would always be a sum of numbers in the Fibonacci sequence. Building on this, Griffiths (2011) also used an approximation for the n^{th} term of the Fibonacci sequence to find an approximation for the number of sequences of n tosses of the coin that contained m pairs of consecutive heads, such that the probability of having m pairs of consecutive heads after n tosses of the coin could also be estimated. While Griffiths (2011) is instructive, its scope is fundamentally different from the scope of the first research question; Griffiths (2011) looks at *multiple, possibly separate, pairs* of consecutive heads, while this project looks at a *single series* of *multiple* consecutive heads. This difference is illustrated by the fact that two pairs of consecutive heads could possibly mean three consecutive heads, but also two distinct pairs of consecutive heads comprising four heads altogether (see Figure 1). Furthermore, Griffiths (2011) used an approximation for terms in the Fibonacci sequence, undermining the accuracy of any probability calculated using it.

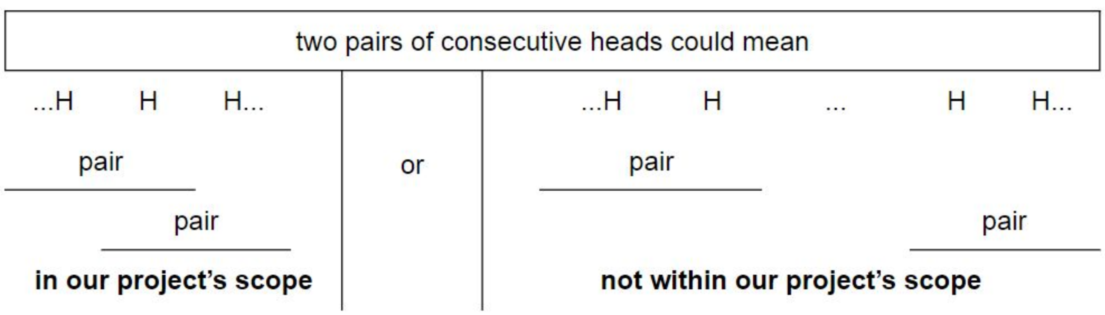


Figure 1: Two pairs of two consecutive heads within the scope of Griffiths (2011)

Finkelstein and Whitley (1978) furthermore also found an expression in terms of m, p, q and the n^{th} term of the Fibonacci sequence for the probability of a sequence of p tosses of a coin having q consecutive heads, where the probability of achieving a single head was m. These results can be useful, especially in answering the second and third research questions, but are limited in their utility by the fact that the scope of Finkelstein and Whitley (1978) is slightly different from that of this project. Finkelstein and Whitley (1978) considers not only sequences of coin-tosses that end with q consecutive heads, but also sequences of coin-tosses that have q consecutive heads at any point in the sequence. This is to say that a sequence of p coin-tosses continuing even after the desired number of consecutive heads has been achieved at the $(p - x)^{\text{th}}$ toss would be accepted by Finkelstein and Whitley (1978), though it would not be within the scope of this project. Work will have to be made on the results of Finkelstein and Whitley (1978) in order to adapt and adjust its results to the scope of this project, in particular the second and third research questions.

Methodology

Outline of Methods

- Mathematical induction, a common technique used in mathematical proofs, can be used to look for a pattern of the values of individual probabilities, so as to retrodict and find an expression of the probability for the general cases described in the research question. Mathematical induction is, at its core, the process of proving that a property $P(n)$ holds for every natural number n by observing the traits of $P(n)$ for smaller values of n, and generalising these traits to all natural numbers n.
- Combinatorics techniques can also be used to find the number of possible desired results in a sequence of n coin throws i.e. sequences of n coin throws that end in

m consecutive heads. Dividing this by the number of possible sequences of n coin throws, which is 2^n, will allow the desired probability function to be found.

- Double counting, a technique commonly used in finding probabilities, can also be used. It may simplify the proof, because all that essentially has to be done is to find multiple ways of calculating the desired probability, and to prove that the multiple ways used result in the same probability function.

Work on Research Questions

RQ1

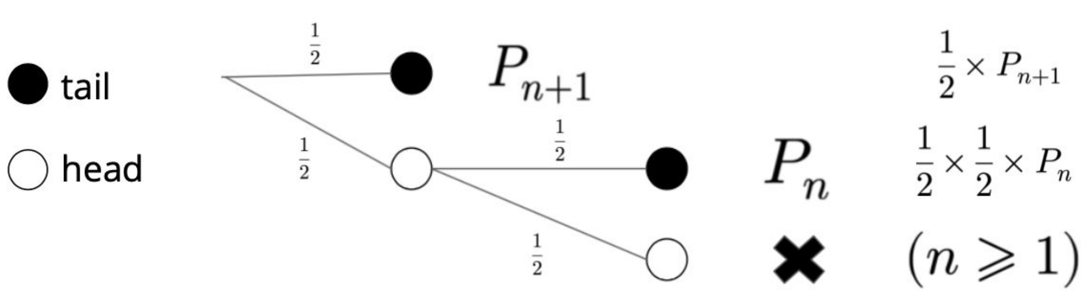
What is the probability that n tosses of a coin are required for m consecutive heads to appear?

First of all, we define that P_n is the probability that n tosses of the coin are required for m consecutive heads to appear. F_n is the number of situations where n tosses of the coin end with m consecutive heads. We can easily get the relationship between P_n and F_n, which is

$$F_n = 2^n P_n \quad .$$

It would be worth beginning from the case of two consecutive heads, when $m = 2$, which is the case in Willard, Dobbs and McQuillan (1986).

With $P_1 = 0$, $P_2 = \frac{1}{4}$, we try to find P_{n+2} where $n > 0$



By observing, we realize that if the first toss is a tail, the desired two consecutive heads must be achieved in the subsequent $n+1$ tosses. If the first toss is a head, and the second a tail, the desired two consecutive heads must be achieved in the subsequent n tosses.

Using the Law of Total Probability, it can be proved that

$$P_{n+2} = \frac{1}{2}P_{n+1} + \frac{1}{4}P_n$$

As a general formula for P_n cannot be directly found, a formula for F_n is found first, given that.

$$P_n = \frac{1}{2}P_{n-1} + \frac{1}{4}P_{n-2} \qquad F_n = 2^n P_n$$

In this case, it becomes apparent that

$$F_n = F_{n-1} + F_{n-2}$$

In this case, F_n is the n^{th} Fibonacci number.

Since the Fibonacci sequence has the characteristic equation $x^2 = x + 1$, whose roots are $x_1 = \frac{1+\sqrt{5}}{2}$ and $x_2 = \frac{1-\sqrt{5}}{2}$, the general formula for the n^{th} Fibonacci number is found to be

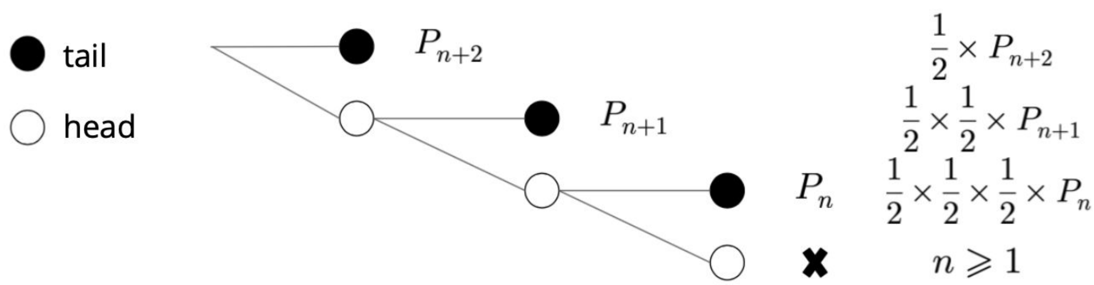
$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$$

Since the number of sequences of n coin-throws that end with two consecutive heads is the n^{th} Fibonacci number, using the general formula for F_n , one can find that where $m = 2$, the probability that n coin throws ends with two consecutive heads is

$$P_n = \frac{1}{2^n\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$$

The method for $m = 3$ is the same as that for $m = 2$

We try to find P_{n+3} where $n > 0$



Using the Law of Total Probability, the relationship between the term P_{n+3} and preceding terms can be obtained.

$$P_{n+3} = \frac{1}{2}P_{n+2} + \frac{1}{4}P_{n+1} + \frac{1}{8}P_n$$

After that, by using the relationship between P_n and F_n ,

$$P_n = \frac{1}{2}P_{n-1} + \frac{1}{4}P_{n-2} + \frac{1}{8}P_{n-3} \qquad F_n = 2^n P_n$$

the expression for F_n is found to be

$$F_n = F_{n-1} + F_{n-2} + F_{n-3}$$

This recurrence relation has the characteristic equation

$$x^3 = x^2 + x + 1$$

The general term of this sequence is

$$F_n = c_1 x_1^n + c_2 x_2^n + c_3 x_3^n$$

where x_1, x_2, x_3 are the three roots of the characteristic equation, and

$$c_1 = \frac{1}{x_1(x_2 - x_1)(x_3 - x_1)}$$

$$c_2 = \frac{1}{x_2(x_1 - x_2)(x_3 - x_2)}$$

$$c_3 = \frac{1}{x_3(x_1 - x_3)(x_2 - x_3)}$$

Using the relationship between P_n and F_n ,

$$F_n = 2^n P_n \qquad F_n = c_1 x_1^n + c_2 x_2^n + c_3 x_3^n$$

one finds an expression for P_n when $m = 3$, which is

$$P_n = \frac{c_1 x_1^n + c_2 x_2^n + c_3 x_3^n}{2^n}$$

The above expression is the probability that *n* coin throws ends with three consecutive heads.

By induction, one observes that F_n can be expressed as the sum of m preceding terms as below

$$F_n = \sum_{j=1}^{m} F_{n-j}$$

such that characteristic equations, as below, can always be used to find the general formula for F_n

$$F_n = \sum_{j=1}^{m} c_j x_j^n$$

The above expression is for the number of sequences of n coin throws that end with m consecutive heads. The method used to find the above expression is described below.

As was done for the case of $m = 2$ and $m = 3$, i.e. the cases where n coin throws ended in two or three consecutive heads, the Law of Total Probability can be applied to obtain the recurrence relationship below

$$P_n = \sum_{j=1}^{m} \frac{1}{2^j} P_{n-j}$$

The recurrence relationship of F_n is then found to be

$$F_n = \sum_{j=1}^{m} F_{n-j}$$

with the characteristic equation

$$x^m = x^{m-1} + x^{m-2} \cdots + x + 1$$

Solving this allows one to find an expression for F_n, for the number of sequences of n coin throws that end with m consecutive heads.

$$F_n = \sum_{j=1}^{m} c_j x_j^n$$

Dividing this by the number of possible sequences of n coin throws, which is 2^n, one can find an expression for the probability that n coin throws are required to obtain m consecutive heads,

$$P_n = \frac{\sum_{j=1}^{m} c_j x_j^n}{2^n}$$

where $x_1, x_2, ..., x_n$ are solutions of the equation:

$$x^m = x^{m-1} + x^{m-2} \cdots + x + 1$$

RQ2

For a game where each round has k outcomes, with each having an equal probability of occurring, where the game ends immediately after the first occurrence of 2 consecutive rounds with identical outcomes, what is the probability that n rounds are required for the game to end?

This game can also be interpreted as a general case of RQ1, where the probability of obtaining a head, the desired outcome, is no longer $\frac{1}{2}$, but instead $\frac{1}{k}$.

The first step is double counting. Whereas the probability of obtaining two consecutive rounds with the desired, identical outcome after n rounds of the game is $P_n,\ n > 1$, P_n can be found by looking at the two possible situations arising after the first round individually. The first situation is where the first round results in the desired outcome, with this situation having a $\frac{1}{k}$ probability of occurring. The second situation is where the first round does not result in the desired outcome, with this situation having a $1 - \frac{1}{k}$ probability of occurring. The latter is illustrated in Figure 2, with the blue part of the circle being the desired outcome, and the orange part the undesired outcome.

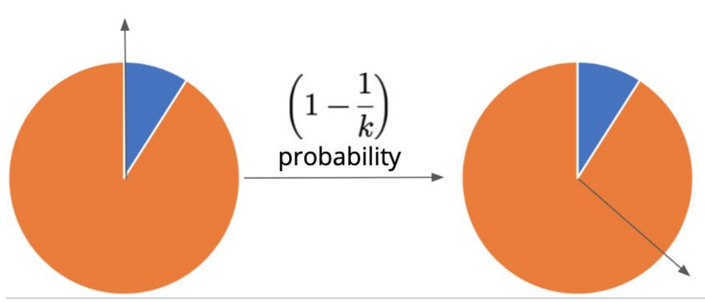


Figure 2: One possible situation arising after the first round of the game

If the outcome of the first round is not the desired one, as shown in Figure 2, P_n is simply equal to the probability of achieving two consecutive rounds with the identical, desired outcome in $n-1$ rounds, since the result of the first round has no impact on that of subsequent rounds. P_n is then equal to

$$\left(1-\frac{1}{k}\right)P_{n-1}$$

On the other hand, if the desired outcome is achieved in the first round, there are two possible situations. The first is where $n=2$, which is to say that the game ends after two consecutive rounds in which an identical, desired outcome is achieved. P_n would then be equal to

$$\frac{1}{k^2}$$

However, where $n>2$, the outcome of the second round must not be the desired one, as illustrated in Figure 3. If the outcome of the second round is the desired one, the game would have ended and n would have been equal to 2, as described above.

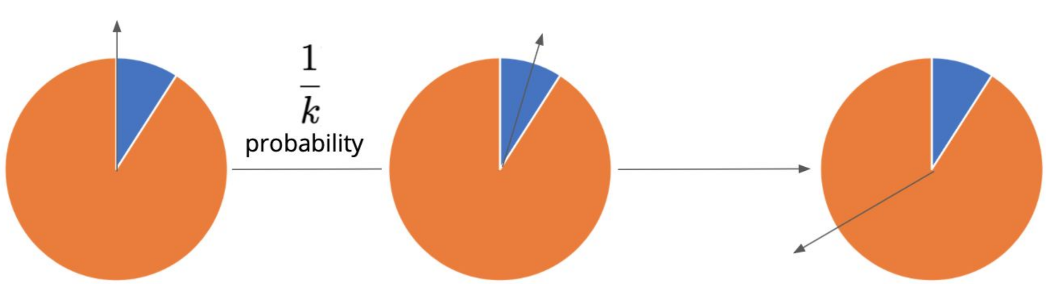


Figure 3: Where the desired outcome is achieved in the first round, and $n>2$, the desired outcome cannot be achieved in the second round

For the situation described in Figure 3, P_n can then be expressed as

$$\left(1-\frac{1}{k}\right)\frac{1}{k}P_{n-2}$$

Using the Law of Total Probability, a recurrence relationship can then be found as below

$$P_n=\left(1-\frac{1}{k}\right)P_{n-1}+\frac{1}{k}(1-\frac{1}{k})P_{n-2}$$

This recurrence equation will again be solved using its characteristic equation, whose validity will be proved as below.

For the general term of recurrence equation which is

$$F_n=a_1F_{n-1}+a_2F_{n-2}$$

its characteristic equation is

$$x^2=a_1x+a_2$$

and its two roots satisfy the following conditions according to Vieta theorem

$$x_1+x_2=a_1$$
$$x_1x_2=-a_2$$

If $x_1 \neq x_2$,

$$F_n=(x_1+x_2)F_{n-1}-x_1x_2F_{n-2}$$

Simplifying the equation, it is found that

$$F_n-x_1F_{n-1}=x_2\left(F_{n-1}-x_1F_{n-2}\right)$$

$$F_n - x_2 F_{n-1} = x_1 \left(F_{n-1} - x_2 F_{n-2} \right)$$

Let

$$P_n = F_n - x_1 F_{n-1}$$

and

$$Q_n = F_n - x_2 F_{n-1}$$

such that it can be found using mathematical induction that

$$P_{n+1} = x_2 P_n \qquad Q_{n+1} = x_1 Q_n$$

$$P_n = x_2 P_{n-1} \qquad Q_n = x_1 Q_{n-1}$$

$$P_{n-1} = x_2 P_{n-2} \qquad Q_{n-1} = x_1 Q_{n-2}$$

$$\ldots \qquad \ldots$$

$$P_2 = x_2 P_1 \qquad Q_2 = x_1 Q_1$$

By multiplying them together, it is found that

$$P_{n+1} = x_2^n P_1 \qquad Q_{n+1} = x_1^n Q_1$$

A set of simultaneous equations is then obtained, as below

$$\begin{cases} P_{n+1} = F_{n+1} - x_1 F_n = x_2^n P_1 \\ Q_{n+1} = F_{n+1} - x_2 F_n = x_1^n Q_1 \end{cases}$$

Solving this set of simultaneous equations gives an expression for F_n as below

$$F_n = \frac{x_1^n Q_1 - x_2^n P_1}{x_1 - x_2}$$

Letting

$$\begin{cases} c_1 = \dfrac{Q_1}{x_1 - x_2} \\ c_2 = \dfrac{P_1}{x_1 - x_2} \end{cases}$$

one obtains the equation

$$F_n = c_1 x_1^n + c_2 x_2^n$$

There is yet another equation for F_n, as below.

$$F_n = a_1 F_{n-1} + a_2 F_{n-2}$$

Combining these two equations gives

$$(c_1 x_1^n + c_2 x_2^n) = a_1 \left(c_1 x_1^{n-1} + c_2 x_2^{n-1}\right) + a_2 \left(c_1 x_1^{n-2} + c_2 x_2^{n-2}\right)$$

which, when simplified, becomes the equation

$$\left(x_1^2 - a_1 x_1 - a_2\right) c_1 x_1^{n-2} + (x_2^2 - a_1 x_2 - a_2) c_2 x_2^{n-2} = 0$$

Whereas

$$\begin{cases} c_1 x_1^{n-2} > 0 \\ c_2 x_2^{n-2} > 0 \end{cases}$$

one obtains the following expressions for x_1 and x_2

$$\begin{cases} x_1^2 - a_1 x_1 - a_2 = 0 \quad (1) \\ x_2^2 - a_1 x_2 - a_2 = 0 \quad (2) \end{cases}$$

x_1 and x_2 being the roots of the equation

$$x^2 = a_1 x + a_2$$

Having found this, given the recurrence relationship for P_n that was previously found,

$$P_n = \left(1 - \frac{1}{k}\right) P_{n-1} + \frac{1}{k}(1 - \frac{1}{k}) P_{n-2}$$

the characteristic equation is found to be

$$x^2 + \left(\frac{1}{k} - 1\right) x + \frac{1}{k}(\frac{1}{k} - 1) = 0$$

such that P_n can be expressed as

$$P_n = k_1 x_1^{n-2} + k_2 x_2^{n-2}$$

When $n = 2$, it is easily found that

$$P_2 = \frac{1}{k^2}$$

When $n = 3$, it is also not difficult to find that

$$P_3 = (1 - \frac{1}{k})(\frac{1}{k^2})$$

By substituting this into our system of equations as below,

$$\frac{1}{k^2} = k_1 + k_2$$

$$\frac{1}{k^2}(1 - \frac{1}{k}) = k_1 x_1 + k_2 x_2$$

one can find that

$$k_1 = k_2 = \frac{1}{2k^2}$$

which, when substituted into the expression for P_n, allows one to find an equation for P_n in terms of k and n, which are defined in the Research Question.

$$P_n = \frac{1}{2k^2}\left[\frac{\left(1 - \frac{1}{k} + \sqrt{\frac{-3}{k^2} + \frac{2}{k} + 1}\right)^{n-2}}{2} + \frac{\left(1 - \frac{1}{k} - \sqrt{\frac{-3}{k^2} + \frac{2}{k} + 1}\right)^{n-2}}{2}\right]$$

This is the probability that n rounds of a game end with two consecutive rounds with an identical outcome, where each round has k possible outcomes.

RQ3

For a game where each round has k outcomes, with each having an equal probability of occurring, where the game ends immediately after the first occurrence of m consecutive rounds with identical outcomes, what is the probability that n rounds are required for the game to end?

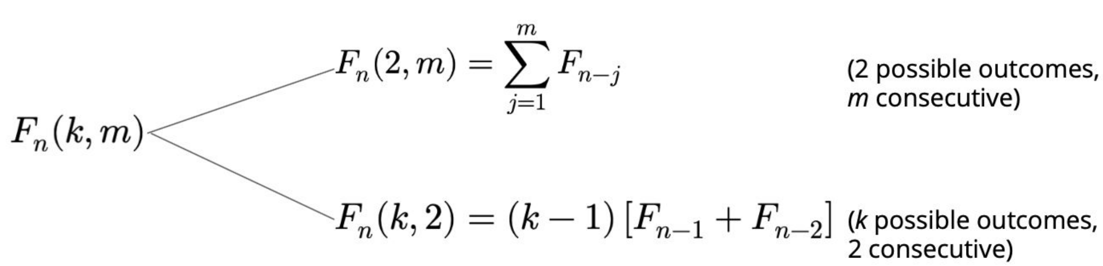
It was found for RQ1 that the probability of achieving m consecutive rounds with an identical outcome at the end of n rounds of a game, where each round had two possible outcomes, was

$$F_n(2,m) = \sum_{j=1}^{m} F_{n-j}$$

It was also found for RQ2 that the probability of achieving two consecutive rounds with an identical outcome at the end of n rounds of a game, where each round had k possible outcomes, was

$$F_n(k,2) = (k-1)\left[F_{n-1} + F_{n-2}\right]$$

By combining them as below



An expression for F_n for m consecutive rounds with an identical outcome in a game where each round had k possible outcomes can be found as below

$$F_n(k,m) = (k-1)\sum_{j=1}^{m} F_{n-j}$$

Its characteristic equation was found to be

$$x^m = (k-1)(x^{m-1} + x^{m-2} + \cdots + x + 1)$$

which, when solved, gave an expression for F_n

$$F_n(k,m) = \sum_{j=1}^{m} c_j x_j^n$$

Dividing this by the number of possible sequences of n rounds of such a game, which is k^n an expression for P_n can be found as below,

$$P_n(k,m) = \frac{\sum_{j=1}^{m} c_j x_j^n}{k^n}$$

Hence, for a game where each round has k outcomes as described previously, where the game ends immediately after the first occurrence of m consecutive rounds with identical outcomes, the probability that n rounds are required for the game to end is

$$P_n(k,m) = \frac{\sum_{j=1}^{m} c_j x_j^n}{k^n}$$

where $x_1,\ x_2, \ldots,\ x_n$ are solutions of the equation:

$$x^m = (k-1)(x^{m-1} + x^{m-2} + \cdots + x + 1)$$

This is the probability that m consecutive rounds with an identical outcome are achieved at the end of n rounds of a game where each round has k possible outcomes.

Conclusion and Discussion

Conclusion

In conclusion, the solutions to our three RQs are as follows:

For RQ1, the probability that n tosses of a coin are required for m consecutive heads to appear is

$$P_n = \frac{\sum_{j=1}^{m} c_j x_j^n}{2^n}$$

where $x_1, x_2, \cdots, x_n$ are the solutions to the equation

$$x^m = x^{m-1} + x^{m-2} \cdots + x + 1$$

For RQ2, where each round has k possible outcomes, with each having an equal probability of occurring, where the game ends immediately after the first occurrence of 2 consecutive rounds with identical outcomes, the probability that n rounds are required for the game to end is

$$P_n = \frac{1}{2k^2}\left[\frac{\left(1-\frac{1}{k}+\sqrt{\frac{-3}{k^2}+\frac{2}{k}+1}\right)^{n-2}}{2}+\frac{\left(1-\frac{1}{k}-\sqrt{\frac{-3}{k^2}+\frac{2}{k}+1}\right)^{n-2}}{2}\right]$$

For RQ3, where each round has k possible outcomes, with each having an equal probability of occurring, where the game ends immediately after the first occurrence of m

consecutive rounds with identical outcomes, the probability that n rounds are required for the game to end is

$$P_n(k,m) = \frac{\sum_{j=1}^{m} c_j x_j^n}{k^n}$$

where $x_1, x_2, \cdots, x_n$ are the solutions to the equation

$$x^m = (k-1)(x^{m-1} + x^{m-2} + \cdots + x + 1)$$

Discussion of Limitations and Recommendations

This section will discuss the limitations of our research with reference to literature, and propose possible extensions of our research.

The arguably greatest limitation of this work is that the expressions of our solutions to our three RQs are not related to the Fibonacci sequence, even though Finkelstein and Whitley (1978) suggested that this was possible, having found an expression in terms of m, p, q and the n^{th} term of the Fibonacci sequence for the probability of sequences of p tosses of a coin having q consecutive heads, where m, a variable, was the probability of achieving a single head. This, of course, assumes that the probability functions for RQ1, 2 and 3 can be expressed and are best expressed using certain terms of the Fibonacci sequence; no other integer sequences were considered in this project as a result. A possible extension would be to relate the probability functions of RQ1, 2 and 3 to the n^{th} term of the Fibonacci sequence and other integer sequences that might better express the probability functions of RQ1, 2 and 3.

Another limitation of this work was the assumption that each of the possible outcomes in a game had an equal probability of occurring. While this is true of most games, not least those using fair coins and fair dice, a possible extension would be to adjust the probability functions obtained for RQ1, 2 and 3 for games whose outcomes do not have an equal probability of occurring, e.g. a weighted dice. This would also extend the applicability of this work to games with multiple desired outcomes, whose probability cannot be expressed as a unit fraction, such as the probability of drawing two court cards in a 52-card deck.

Last but not least, this work assumes that the result of each game is independent of the results of previous rounds. This is certainly not true of all games. The probability functions obtained for RQ1, 2 and 3 should be adjusted to extend the applicability of this work to such games where the result of each round is a dependent event i.e. is affected by the results of previous rounds by considering conditional probabilities. The results of this work would then be more universally applicable, not least to games such as drawing spades from a deck of cards.

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