# Lights Out!

**ID: 8-10**

Written report by

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## Abstract

This project is about the mathematical game, Lights Out puzzle, where the player plays using an electronic device. He has to turn a grid from one colour to another by selecting squares, which causes its 4 adjacent squares and itself to change colour. There are only 2 colours and the square changes back when selected/affected again.

This project aims to find a method to solve the puzzle in the least number of moves, and explore its variations. The variations are based on the original puzzle but have changes made to make it more challenging. We used algebra and number theory to find a method for the original puzzle.

We have found that it is possible to use algebra to represent each square. We let the top row of the grid be unknowns, and then deduce downwards. We can thus represent the rest of the grids using the unknowns, and form algebraic equations. The number of equations formed is always equal to the number of unknowns, so we can definitely solve it using this method.

As for the variations, we managed to solve one of them, Lights Out X, by separating and transforming it into basic puzzles. This allows us to solve it as we have already found a method to solve the basic puzzles.

We hope, in doing this, increases the popularity of these mathematical puzzles.

## Contents

# Introduction

## Description

The Lights Out puzzle was first invented by Tiger Electronics. It is a mathematical puzzle using a square grid. It is usually done on electronic devices such as laptops and mobile phones, although the original version had its own device. The puzzle is usually a 5 x 5 grid with squares all grey.

When you start the game, all the lights are on (grey), and you have to solve it by “turning off” all the lights, and our group will be exploring with different square board sizes. We will be exploring methods that solve these puzzles with minimum steps, and also trying out some variations like the Three Colour Puzzle and Lights Out puzzle X.

The puzzles we will work on start with a grid, all grey (lit up). It can be represented in a matrix like this 3x3:

0 0 0
0 0 0
0 0 0

Where 0 represents lit and 1 represents unlit. What we want to achieve is:

1 1 1
1 1 1
1 1 1

How we “turn” the lights off is by selecting squares, one by one. The selected square, along with its 4 adjacent neighbours, will switch its state. When it is 0, or grey in colour, it turns to 1, or green in colour, and vice versa. So if I select the middle square of the 3x3, it becomes:

0 1 0
1 1 1
0 1 0

This game is similar to other mathematical games, like the Rubik's cube. The aim is to construct without destroying what you have constructed.

There are also variations to the puzzle. One that we will explore is the Lights Out X. The setup and aim of the game is still the same. However, when you select a square, instead of its directly adjacent squares toggling, its diagonally adjacent ones will. For example, when we select the middle square of the 3x3, it will look like this:

1 0 1
0 1 0
1 0 1

Another variation is that it has 3 colour codings. When grey, or 0, turns into green, or 1, it will turn into another colour, red, or 2, when selected again. After that, it will turn back into grey when selected/affected by another square.

## Objectives

We want to:

- Experiment on the Lights Out puzzles
- Find a method to solve for the most efficient way to solve the Lights Out puzzle, no matter the size
- Explore the variations of the puzzle and come up with a method to solve them as well, in the least possible moves if possible.

Some questions we ask are:

- How do we know if a solution is the shortest?
- Can there be more than one solution?
- Is there a relationship between the normal puzzle and its variations?

# Literature Review

## Original Lights Out

There has been plenty of research done on the original Lights Out Puzzle, but *none* on its variations, Lights Out X and 3 colour codings.

Sutner (1989) had stated and proved that “going from all lights on to all lights off is always possible for *any* size square lattice”. Barile, Margherita(2002) had made observations and found the solutions of the puzzle up to 7 x 7, but we aim to find a method to solve all square puzzles.

Mathematician Rafael Losada, who wrote an article regarding the puzzle for the SUMA magazine, had come up with a solution using linear algebra. However, the solution and explanation is very complicated and we plan to make a simpler solution.

## Lights Out Variations

There has been no research done on the variations, and thus we plan to explore them and come up with a method of solution. We also want to explore the similarities between them and the original puzzle and how they relate to each other.

# Methodology

## Past Methods

The traditional way of solving the lights out puzzle (even those with all lights lighted at the start) is the Chase The Light method. This requires a list of patterns from the internet. The player

- Press ALL the tiles under the lighted tiles of the first row, in order to light up the entire first row
- Move on to the second row and repeat the process till the last row, where not all the lights are lit up
- Based on the bottom row lights pattern, look through a list of patterns and press the corresponding squares on the first row
- Re-chase the lights.

This will end up with all lights out but this method, unfortunately, is not the best method. It does not provide the most efficient solution and requires an already-made list.

Rafael Losada made a method involving representing each square with algebraic equations and using imaginary squares. Our method is somewhat similar, but simpler.

## Observations (Original Puzzle)

- Each square in the original puzzle has only two states, lit or unlit.
- It is pointless to click the same square twice as It has the same effect as before.
- A square can only change its state if you pressed the cube at the top,below, left, right or itself.

## Solution (Original Puzzle)

We will use the 3x3 as an example.

- We will name the top 3 squares **a**,**b** and **c** and the value they take is 0 or 1 for whether it has been pressed. 1 represents selected and 0 represents not selected. The rest of the grid are as follows:

**a b c**
**d e f**
**g h i**

- We start from the letter **a**. The value **a** is either 0 or 1, and **a** must be illuminated. This means that **a**+**b**+**d** must be an odd number. (You can only press all three of the squares or only one square to light up **a**)

It can be expressed as: **a**+**b**+**d**=1(mod 2)

This notation can be simplified if we agree that, from now on, all the mathematical operations will be done with the algebra modulo 2 (where 1+1=0).

We switch **d** and 1 and it becomes: **a**+**b-1**=**-d**, but since negative is the same as positive in mod 2, **a**+**b**+1=**d**. Now, we can replace **d** with this equation. Using the same method, we can form **a**+**b**+**c**+**e**=1 by looking at **b**. Thus, **a**+**b**+**c**+1=**e** and we can replace **e**.

Sometimes, we can simplify the equation. Since it is in mod 2, anything even is equivalent to 0. For example, 2**a**+3**b**+5 is the same as **b**+1, as we can subtract 2**a**+2**b**+4 from it.

We follow the same steps for the second row, and in the end we get the grid:

| | | |
|---|---|---|
| a | b | c |
| a+b+1 | a+b+c+1 | b+c+1 |
| a+c+1 | 0 | a+c+1 |

Now, we form some equations to solve. To light up the third row 1st square, we need:
(**a**+**b**+1)+(**a**+**c**+1)+(0)=1
Simplified to:
**b**+**c**=1
Looking at the 2nd and 3rd square, we form:
**a**+**b**+**c**=0 and **a**+**b**=1 (both simplified) respectively.

Thus, there are 3 unknowns and 3 equations, which are **b**+**c**=1, **a**+**b**+**c**=0 and **a**+**b**=1.

We can now solve the equations.
From **b**+**c**=1 and **a**+**b**+**c**=0, we can see that **a**=1. Looking at **a**+**b**=1, we now get **b**=0. Since **b**+**c**=1, **c**=1.

And voila! We have found a,b and c. Since the rest of the squares are represented by a, b and c, we have solved the puzzle.

This method works for all square or rectangular puzzles as the number of unknowns is equal to the number of equations formed at the end. Sometimes, there might be more than one solution. Thus, we have found the method to solve the Lights Out Puzzle!

## Observations (Lights Out X)

We noticed that:

- We can tilt the puzzle 45˚ to make it similar to the normal puzzle
- The lights out X puzzle can be separated in a checkerboard pattern, as both sections cannot affect each other in any way.

## Solution (Lights Out X)

Thus, we can

1. tilt the grid
2. separate it into two areas like the black and white on a checkerboard
3. convert each part to the normal lights out, since it will behave the same way.

A 3x3 can be separated as shown:

```
      0                                0
   0     0                                             0     0
0     0     0   becomes:   0     0     0       +
   0     0                                             0     0
```

0 0

The parts now work exactly like a normal puzzle, although not square. We then use algebra like before to solve the two different parts, and get the solutions like the normal puzzle. We can then convert them back to the black and white arrangement and combine the solutions. Example, if there were 3 solutions for the white side and 4 solutions for the black side there would be a total of 3x4=12 solutions.

Here is a worked example of a 5x5:

(Part 1)

c

b c+1 d

a b+c b+d d+c e

a+b+c+d+1 c b+c+d+e+1

c+1

We will get c=1, a=b, d=e, b+d=1 from part 1.

(part 2)

a b

c a+b+1 a+b+1 d

a+b+c c+a+1 d+b+1 a+b+d

a+b+d a+b+c

We will get b+c=0, a+d=0 from part 2. As long as we follow the rules of these equations, we will get the answer. We then combine part 1 and 2 together to form the solution.

# Conclusion

We have:

- Found the best solution to solve the basic lights out puzzle in the least number of moves, for every square grid.
- Solved the lights out X puzzle, using algebra and converting it into normal puzzles..

In the future, we can try:

- the three colour coding
- find a mathematical proof to show our method uses the least number of moves.

## References


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