Dominating Domineering

Written Report

Group ID: 8-05

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Abstract

In this paper we build on data from previous researchers to examine different boards and find lemmas using patterns to predict who wins, regardless of size and shape of the board. We assign integers to the boards such that we can add them together and find the value of the board we pieced together, or split bigger boards into smaller ones and find the value of the original one. We also classify moves and sort them according to importance, to determine who has an advantage from the start. Some key findings include that for any 3 x n board where $n>3$, it is a win for the horizontal player, and for any 2 x n, where $n>27$, it is also a win for the horizontal player. We also note that any n x n board is either a first or second player win, whilst m x n boards where $m>n$ tend to be more advantageous to vertical vice versa.

Contents:

1. Introduction

1.1 What is Domineering?

Domineering is a two-player mathematical game, which can be played on a chessboard or a piece of graph paper. On the graph paper, any design can be traced out to play the game. For example, it can be played a rectangle or it could be played on an irregular polygon.

In Domineering, 2 players have access to 2×1 pieces/dominoes. Player 1 can only place these pieces vertically on the board, while player 2 can only place the piece horizontally on a m x n board. Both players take turns to place their pieces. The first player which is not able to place down another piece, loses the game.

1.2 Domineering game example

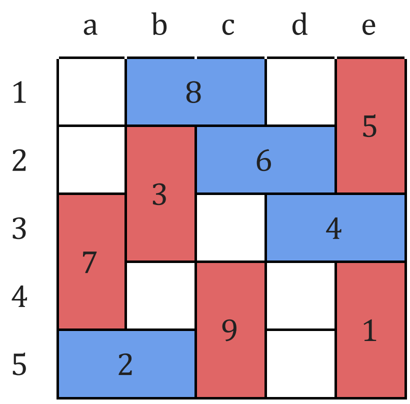


Fig.1 an example of a game of Domineering.
In this case, Red wins since blue is unable to play anymore of his pieces

2. Objectives

- Find a solution to common board sizes of Domineering
- Be able to analyse most situations in Domineering regardless oftheir size or shape
- Find patterns on m x n boards

3. Terminology

3.1 Terms used in combinatorial games

Left (L): Vertical player

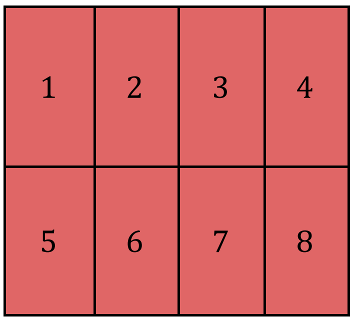
Right (R): Horizontal player

N: First player

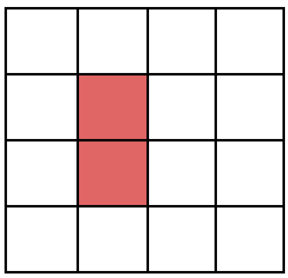
P: Second player

m x n board: a board with m squares height and n squares width

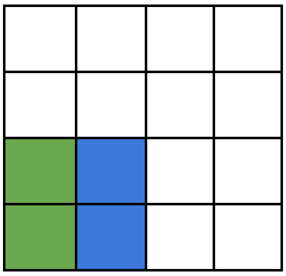
3.2 Terms used in game (Domineering)



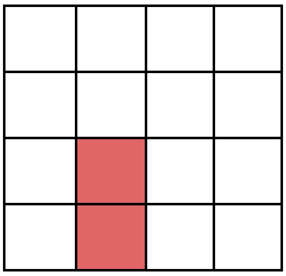
Real Moves: The maximum number of moves a player can make at a certain time if his opponent does not make anymore moves (For example, L has 8 real moves here)



Double-destroying Move: A move that removes 2 of the opponent's real moves. (For example, this move reduces R’s real moves by 2, making it a double-destroying move for L)



Safe Move: A move that cannot be reached by the opponent. (Blue-coloured tiles represent dominos that are already placed, while green tiles are those that cannot be occupied by R no matter how he moves)



Safe-making Move: A move that creates a safe move. (For example, the move here makes the two tiles on its left become untouchable by R, where only L can place a domino, making it a safe move for L)

4. Literature Review

4.1 Perfectly Solving Domineering Boards

Jos W. H. M. Uiterwijk (2014) studied on how to "perfectly solve" Domineering boards, meaning solving them without any search. He achieved this by defining different move types, then coming up with 12 knowledge rules, 6 to prove that Left can win against any opposition, and 6 to prove otherwise. Some of the moves defined are safe moves, double destroying moves, extend double destroying moves and safe making moves, some of which we used to aid us. His knowledge rules are of increasing difficulty, and are arrived from different move types and board parameters, such as RealCur (real moves), DDCur (double destroying moves), SafeCur (safe moves). With this, he managed to perfectly solve 67 out of 81 boards ranging from 1x1 to 10x10, excluding single-row-or-column boards.

4.2 Alpha-beta pruning (brute force)

Alpha-beta pruning is an algorithm to search for moves that are not beneficial to the player and prunes (removes) them from the searching process. This way, we are able to narrow down the total number moves a player can make, making it easier to find the best move the player can make. Alpha-beta pruning is also a good approach as it not only looks at the current board, it also prunes moves by looking a few steps in advance. This further reduces the number of moves we have to specifically look at. Basically, it prunes away moves which can not possibly affect the final result.

The algorithm maintains two values, alpha and beta, which denote the minimum score that the maximizing player is assured of and the maximum score that the minimizing player is assured of respectively. Initially, alpha is negative infinity and beta is positive infinity, both players start with the worst possible score. Whenever the maximum score that the beta player is assured of becomes less than the minimum score that the alpha player is assured of (i.e. beta < alpha), the maximizing player need not consider further descendants of this node, as they will never be reached in the actual play.

History of alpha-beta pruning:

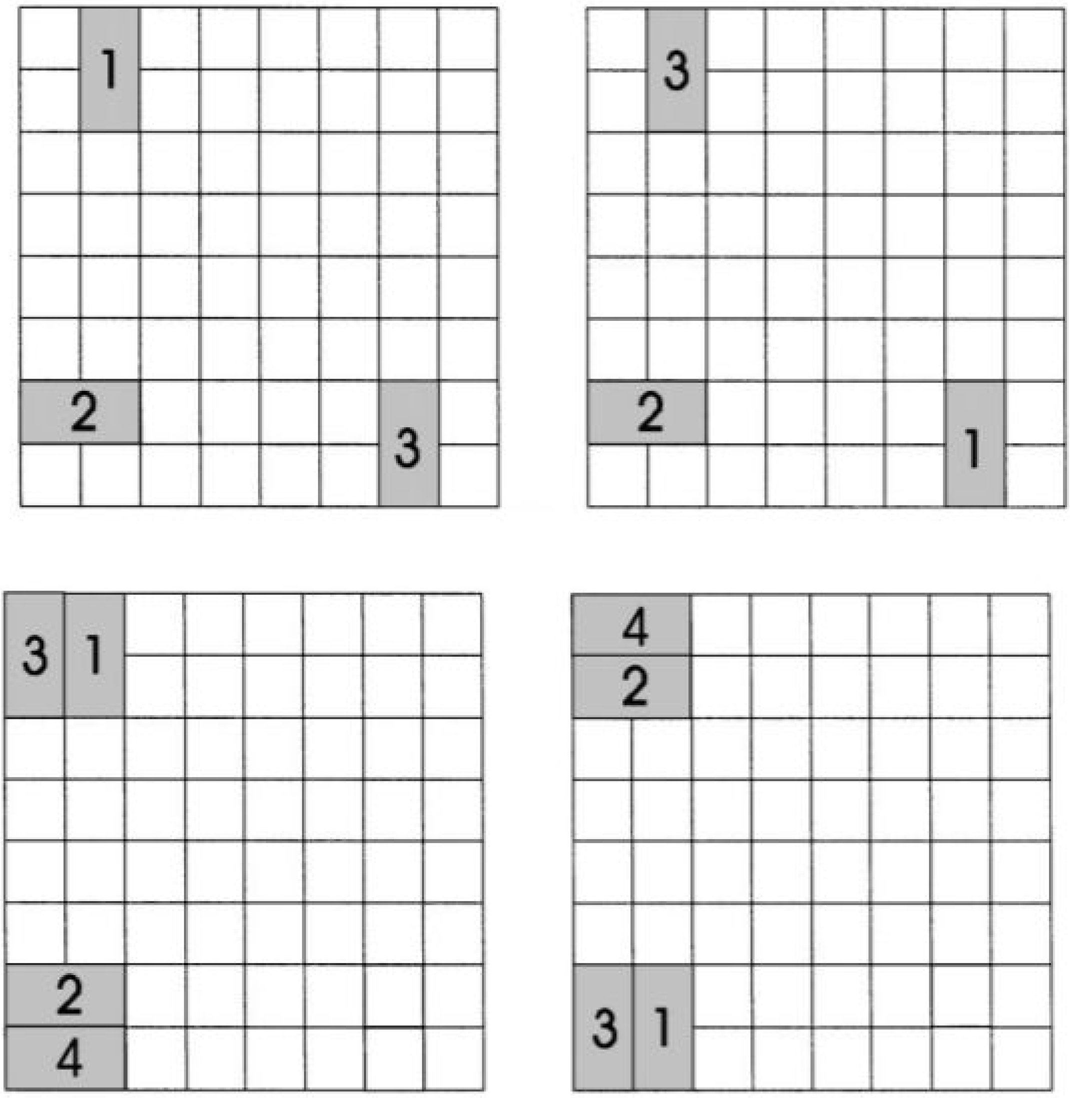
1963: Alexander Brudno published his findings on this topic

1975: Donald Knuth and Ronald W. Moore further refined this algorithm

1986: Michael Saks and Avi Wigderson showed the optimal and randomised version of alpha-beta pruning

4.3 Solving 8x8 Domineering (D.M. Breuker, Jos W.H.M. Uiterwijk, H.J. van den Herik, 2000)

D.M. Breuker, Jos W.H.M. Uiterwijk, H.J. van den Herik (2000) used the technique of transposition tables to solve a 8 x 8 board in Domineering. The result was a first player win. He also applied this technique to m x n boards where 2≤m≤8 and m≤n≤9. They also used the concepts of mobility, real moves and safe moves, where mobility refers to the number of places a player can place a domino in the current situation. Transpositions are defined to be situations with the same outcome although different moves are used to get there.



Some examples of transpositions

To solve the board, they utilised alpha beta pruning, as well as a transposition table to obtain the result for an 8 x 8 board.

5. Solutions to research questions

5.1 Solution to research question 1

RQ 1: For a certain area, how do we determine who would win the given area?

Possible outcomes of any board (predetermined by board size and shape):

- L: Left player win
- R: Right player win
- N: First player win
- P: Second player win

5.1.1 Assigning values to boards:

Rationale:

To enable us to add boards together and be able to determine who would win using the value we found

Values:

Left wins -> Positive

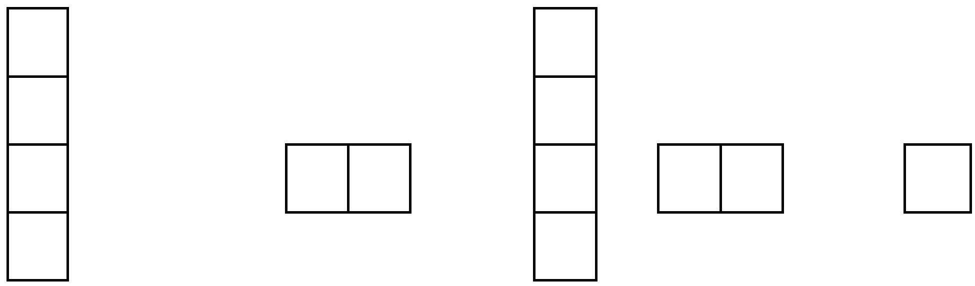
Right wins -> Negative

Second Player wins (P) ->Zero (since both players play equal number of moves)

First Player wins (N) -> Fuzzy (undefined since sometimes Left makes more moves, but sometimes Right makes more moves)

Examples:

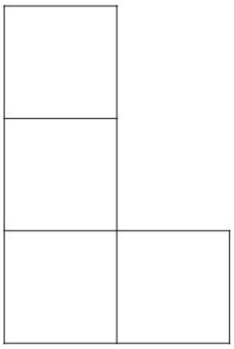
4 x 1 board	1 x 2 board	4 x 1 and 1 x 2 board	1 x 1 board



Value: 2	Value: -1	Value: 2+(-1)=1	Value: 0
Left can place a maximum of 2 dominoes here, while Right cannot place any.	Right can place a single domino in this area, while Left can't place any.	When played optimally, Left can make one more move than Right in the above area.	Neither Left nor Right can make a move, thus the first player loses.

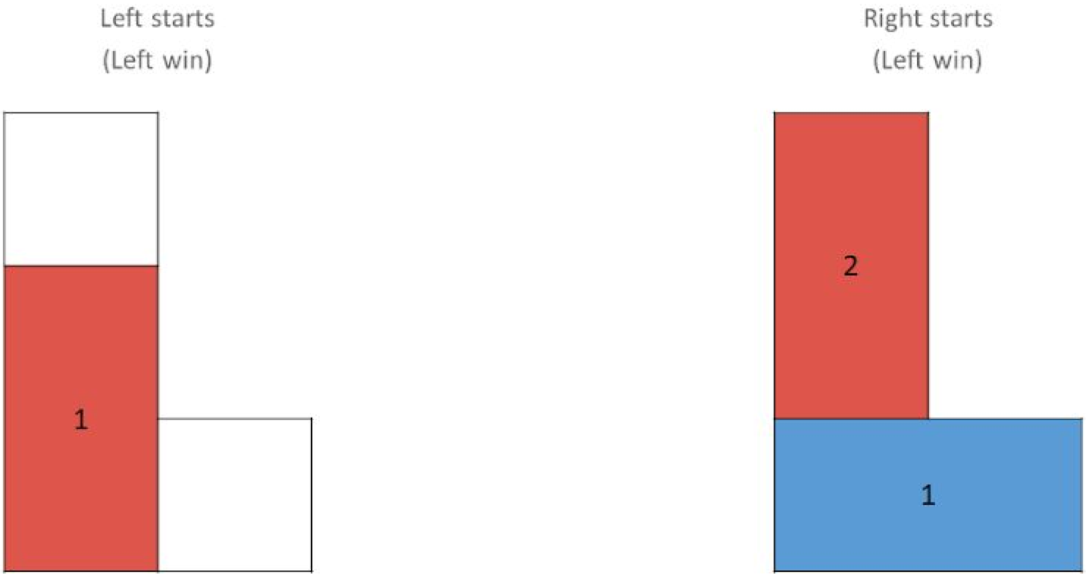
5.1.2 Calculating value of an area:

We start with a board of unknown value.

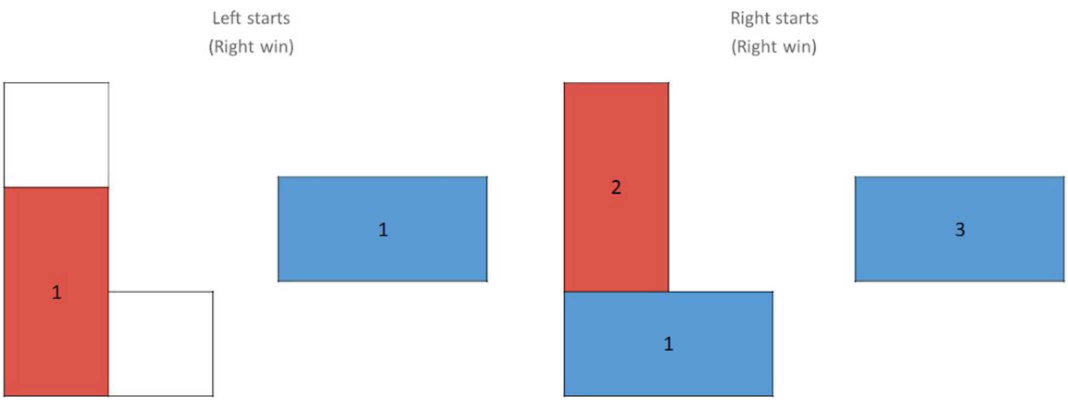


Let the value of the board be n

Check the type of the board (L, R, N, P)

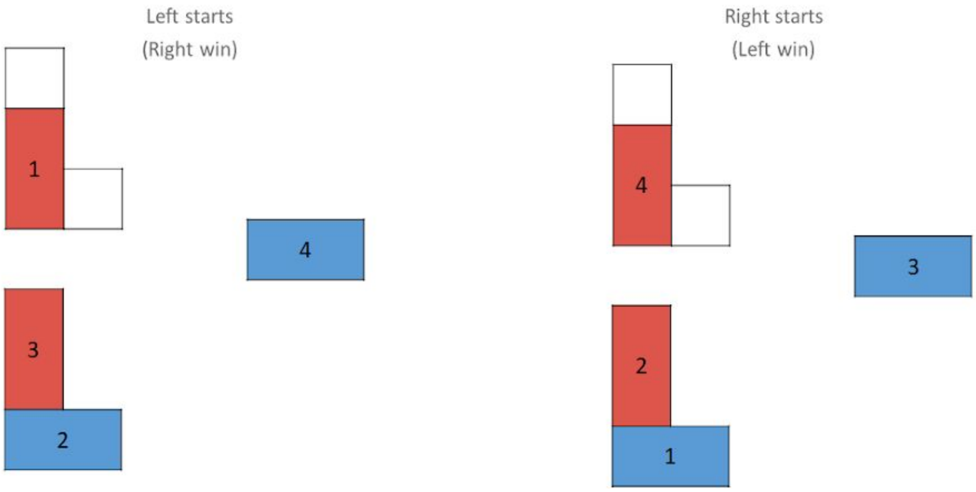


Regardless of the starting player, Left wins, thus this is an L board and $n>0$.
We find the relation between n and 1 by adding a -1 board



Regardless of who starts, Right wins thus $n-1<0$ and $n<1$

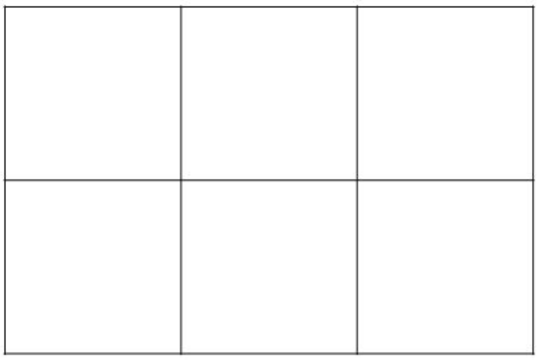
We now know that $0<n<1$, so we check if $2n<1$



2n-1 is a second player win, where both players play an equal number of dominos, thus it is a P board of 0 value. We can conclude that n=½

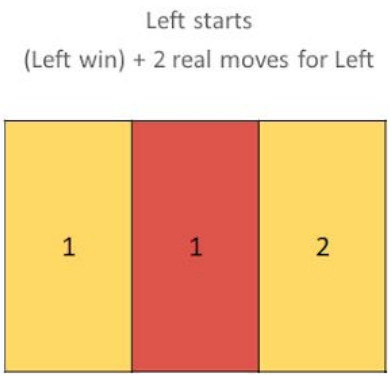
However, this approach does not work if the board is a N board, since it is sometimes positive and sometimes negative, and the absolute value may differ so we use a different approach

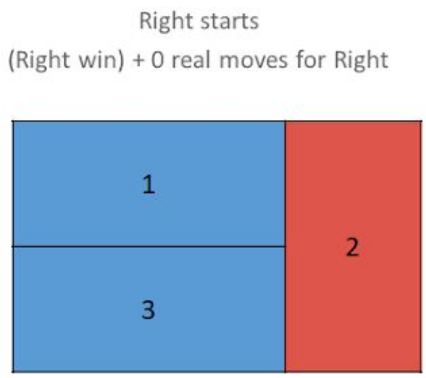
Let's say we have a first-player win board



We first find the outcome assuming both players play optimally

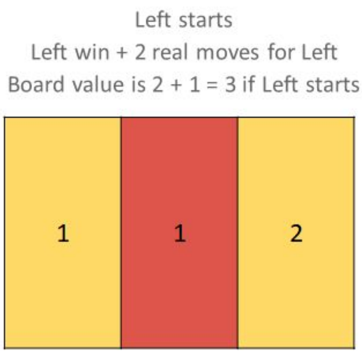
Then we count the number of real moves each player has

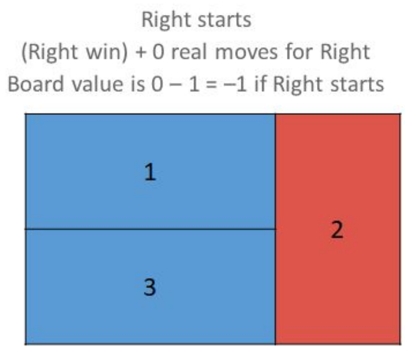




Here, Left has 2 real moves if he starts and Right has no real moves if he starts.

The value of the board is determined by the number of real moves $\pm$ 1.(depending on who wins)





In this case, when Left starts the board value is 2+1=3 and when Right starts it is 0-1=-1

5.1.3 Splitting

Now that we know how to calculate the values of small boards, we can employ the method of splitting on bigger boards:

- Splitting can be done along any part of the board if it is not along optimal moves
- Bad moves:
  - Increases the opponent's safe-making moves/double destroying moves but does not change yours
  - Decreases your safe-making moves/double destroying moves but does not change your opponent's
  - Changes difference in moves between you and opponent, in opponent's favour
- Optimal moves:
  - Moves that minimizes the "badness"
  - Or changes the difference between your real/safe moves and opponent's real/safe moves in your favour

Sometimes we may have to choose between moves, so we have sorted the moves according to priority:

- Safe Moves (Best)
  - Moves that cannot be touched by opponent
- Safe-Making Moves
  - Creates one/two safe move
  - Double-Safe making moves>Single-Safe making moves
- Double-destroying moves
  - Destroys two real moves
- Real moves (Least important)
  - Maximum number of moves that can be played if opponent does not move anymore

To calculate large values of boards:

- We can use splitting of the boards to divide the large board into smaller sub-boards, before summing up the values
- For N boards with fuzzy values, we assign each sub-board with an effective value, which can be defined as no. of real moves opponent gets if he wins + no. of real moves you can get if you win
- We sum them up because when you win the board, you prevent your opponent from getting more safe moves whilst gaining points yourself
- Next we sort all the first player sub-boards according to their effective value and give the odd indexed ones to the first player and even indexed ones to second player
- Repeat the process but swap the first and second player
  - positive for both scenarios: L
  - negative for both scenarios: R
  - for both scenarios it is a zero: P
  - else if it is positive for one but negative for the other:
  - positive when Left starts but zero when Right starts: L and vice versa

5.2 Solution to research question 2

RQ 2: What is the general strategy for 3 x n and 2 x n boards?

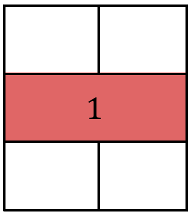
5.2.1 3 x n boards:

Case 1: 3 x 1 board

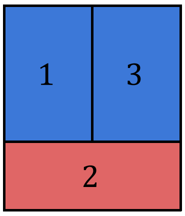
Result: L win

Reason: R has 0 real moves

Case 2: 3 x 2 board
Result: N win
Reason: If L starts, anymove L plays is a winning move.
If R starts, R plays his move in the middle row, reducing L's number of real moves to 0.



If Left goes first



If Right goes first

Case 3: 3 x 3 board
Result: N win
Reason: Either player who starts first can place a domino in the middle tile, effectively leaving his opponent with 1 real move and keeping two safe moves for himself

Case 4: 3 x 4 board
Result: R win
Reason: Regardless of who starts, Right just has to place a domino in the center row to gain 2 safe moves, leaving a 2 x 3 board which at worst gives Left an extra move

Case 5: 3 x 5
Result: R win
Reason: Regardless of who starts, Right can place 1 domino in the center row to gain 2 safe moves. Left can now place a maximum of 3 moves left. On Right's next move, he can place a domino on a spot which is not a safe move for him in order to win.

Case 6: 3 x 6
Result: R win
Reason: If Right starts first, he can make 2 moves in the center row, giving him 4 safe moves. Left can only place a maximum of 2 moves.
If Left starts first, Right can imagine the game as 2 "3 x 3" sub-boards, mirroring Left's move in the other sub-board, rotated 90°.

Case 7: 3 x 7
Result: R win
Reason: Regardless of who starts, Right can place 2 dominoes in the center row, giving him 4 safe moves. Left has at most 3 moves remaining, so Right wins.

Results for 3 x n for n≤7

Board	3 x 1	3 x 2	3 x 3	3 x 4	3 x 5	3 x 6	3 x 7
Result	L	N	N	R	R	R	R

Case 8: 3 x n boards (n≥8)
Result: R win
Reason: R can imagine 3x8 and larger boards as a few different sub-boards of 3x4 to 3x7 boards together.
E.g. 3 x 11 board can be split into 1 3 x 4 and 1 3 x 7 boards
We can do this because L is unable to play his move in 2 different sub boards. R would always have a winning strategy no matter what L plays, since 3 x 4, 3 x 5, 3 x 6 and 3 x 7 boards are all R.

5.2.2 2 x n boards:

Same method of checking individual cases as 3 x n used

Board	2 x 1	2 x 2	2 x 3	2 x 4	2 x 5	2 x 6	2 x 7	2 x 8	2 x 9	2 x 10
Result	L	N	N	R	L	N	N	R	L	N

Board	2 x 11	2 x 12	2 x 13	2 x 14	2 x 15	2 x 16	2 x 17	2 x 18	2 x 19	2 x 20
Result	N	R	P	N	N	R	R	N	N	R

Lemma 1: When n ≡ 0 (mod 4), result is R win
Proof: 2 x 4 is a R win. Whenever L plays his piece, he is playing his piece in a 2 x 4 sub game. R will always be able to counter L with a suitable move. Thus, R always win

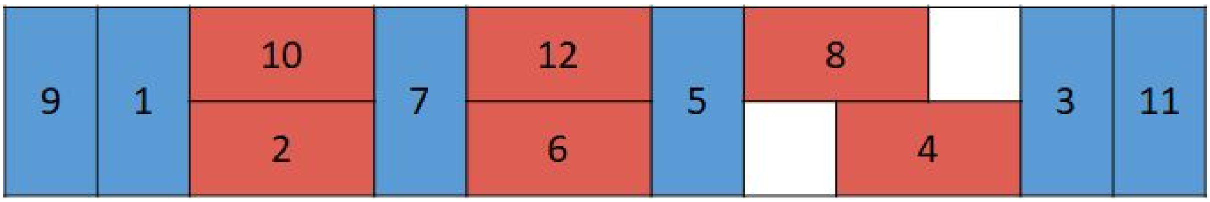
Lemma 2: When n ≡ 2 (mod 4), result is R or N win.
Proof: If R starts, he can see the board as k 2 x 4 sub boards, where k is a positive integer, and 1 2 x 2 sub board.
If R plays his move in the 2 x 2 sub game, L will be forced to play in 1 of the 2 x 4 sub games
Hence, by lemma 1, L will lose if R starts

Lemma 3: When n ≡ 3 (mod 4), result is R or N win.
Proof: If R starts, he can see the board as k 2 x 4 sub boards, where k is a positive integer, and 1 2 x 3 sub board.
If R plays his move in the 2 x 3 sub game, L and R would have 1 real move each in the 2 x 3 sub game. Whenever L does not play his move in the 2 x 3 game, he is

forced to play his move in 1 of the 2 x 4 sub games. Hence, by lemma 1, L would lose if R starts.

2 x 13 board:
Result: P win
Reason: If L goes first, he would place his domino in the second or second last column. R would then make a move adjacent to L's domino to prevent any extra safe making moves from L. L has at most 5 moves remaining while R has at least 2 safe making moves remaining, thus R would win. (2x2+1=5)

If R goes first, any move is a safe making move. L would then place a domino in the second row or second last row, whichever is unoccupied. R can make a maximum of 5 moves after each player has moved once. L has another safe making move after R places his second domino.If R were to be able to place the maximum number of moves, L would also have 5 moves L before R places his second move, thus L wins.



Lemma 4: When n can be expressed as 4a+13b, for all a>0 and b≥0, result is Right win.
Proof: 2 x 13 board is the only board with result P win and 2 x 4 is the smallest 2 x n board with result R win. L cannot cross over different sub 2 x n boards so R can just mirror the sub-boards he plays in. R can get negative values for 2x4 and no change for 2x13 boards. From Q1, the overall value is negative.

When n≥40:
Result: R win
Reason: For any n≥40, n can be expressed as 4a+13b for a>0 and b≥0.

When n=40, n=4 x 10
To obtain n+1, we remove 3 '4's and add 1 '13'
Hence, by lemma 4, R wins when n≥40

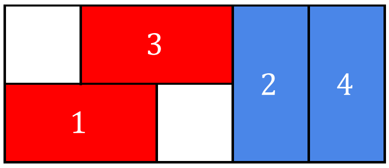
Remaining cases to tackle manually: n=1, 2, 3, 5, 6, 7,9, 14, 15, 18, 19, 22, 23, 26, 27, 31, 35

n=1: L, trivial

n=2: N, trivial

n=3: N, same as 3 x 2

n=5: L, if Left starts, he can place his domino on the second or second last column, giving himself a safe move and reducing Right's real moves to 2. Left's next move has to be somewhere outside his safe move for him to win. If Right starts, he can also make a maximum of 2 moves, while Left has a minimum of 2 moves, resulting in a Left win.



n=6: N, if Left starts first, he will have 2 safe making moves, giving him the win. By Lemma 2, Right can win if he starts.

n=7: N, similar to 2 x 6, if Left starts, he will have 2 safe making moves while Right can only place a maximum of 2 moves. By Lemma 3, Right can win going first.

n=9: L, a winning first move for L is the second or second last column, since it will leave a 2 x 7 which is a -1 for R but it is unable to offset the +2 from L's 2x2 by placing a domino in the second/second-last column. Thus, the total value is +1, making it an L board

n=14: N, by Lemma 2, result is R or N win. If L starts, he can play his piece in the 1st column or last. The remaining 2 x 13 game is P. Since R starts the 2 x 13 game, L would win. Hence, the result is N.

n=15: N, by Lemma 3, Right has a winning strategy going first. If Left starts, he can play his piece in the 2nd column, leaving a 2 x 1 game and a 2 x 13 game. Since R is forced to play in the 2 x 13 game, L would win. Hence, the result is N.

n=18: N, if Left goes first, he can place his domino on the second column, giving him a safe move. Regardless of where Right moves, Left can force it into a 2 x 13 board by moving in the 5th or 16th column. The resulting board is a "2 x 1", 1 "2 x 2" and a "2 x 13". With the previous move, Left ensures that Right's domino is not in the 2 x 2 square., Left can occupy the other one. Result is 1 - 1 + 1 +0 = 1 (Left win). If Right's move is in the 2 x 13 sub-board, Left will play in the 2 x 13 sub-board until Right decides to occupy a 2 x 2 square. In that case, Left will occupy the other one, leading to the same result.

n=19: N, by Lemma 3, Right has a winning strategy going first. If L goes first, he just needs to place his piece in the 10th column or the middle to create 2 2 x 9 sub games. Since 2 x 9 is L win, L wins. Hence, the result is N.

n=22: R, the game can be split into 1 2 x 2 sub-board, and 5 2 x 4 sub-boards. In the best case scenario for Left, Left takes the 2 x 2 sub-board first and gets a +2, for the 2 x 4 sub-boards if L starts the value of the board is 0 and if R starts the value of the board is -1. Since Left already took the 2 x 2 sub-board, it is now Right's turn, and since there is an odd number of 2 x 4 boards, there would be 3 '-1's and 2 '0's. Add that together with the +2 from Left's 2 x 2 sub-board and at best the value of the board is -1 if Left starts. If Right starts, the value will be -4, since Right can take the 2 x 2 sub-board, leaving Left to start, resulting in 2 '-1's for the 2 x 4 sub-boards, and 3 '0's.

n=23: N, by Lemma 3, Right has a winning strategy going first. If L starts, he can play his piece in the 10th column, splitting the board into a 2 x 10 and a 2 x13 board, which L and P win respectively. Whatever, R plays, L has a move to counter it. Hence L wins and the result is N.

n=26: R, if Right imagines it as 2 "2 x 13" subgames, he will have a winning strategy going second. By Lemma 2, Right can win going first. This is the same for

n=27: N, by Lemma 3, Right has a winning strategy going first. If L starts, he can play his move in the middle column, creating 2 2 x 13 boards. Since 2 x 13 is P win, L can win. Hence, the result is N.

n=31: R, Right has a winning strategy regardless of who starts. Left is unable to cross the sub-boards and thus Right can visualise it as a 2 x 9 and 2 x 22, with the values +1 and -4 if Right starts, and -1 if Left starts respectively. If Left starts, the board is of value 0 making it a second player win (Right) board, and when Right starts the value of the board is 1+(-4)=-3, meaning Right wins too.

n=35: R, Right can see it as a 2 x 13 and 2 x 22 subgame, which are P and R respectively, thus Right has a winning strategy.

5.3 Solution to research question 3

Relation between two numbers m and n:

- m is bigger than n
- m equals to n
- m is smaller than n

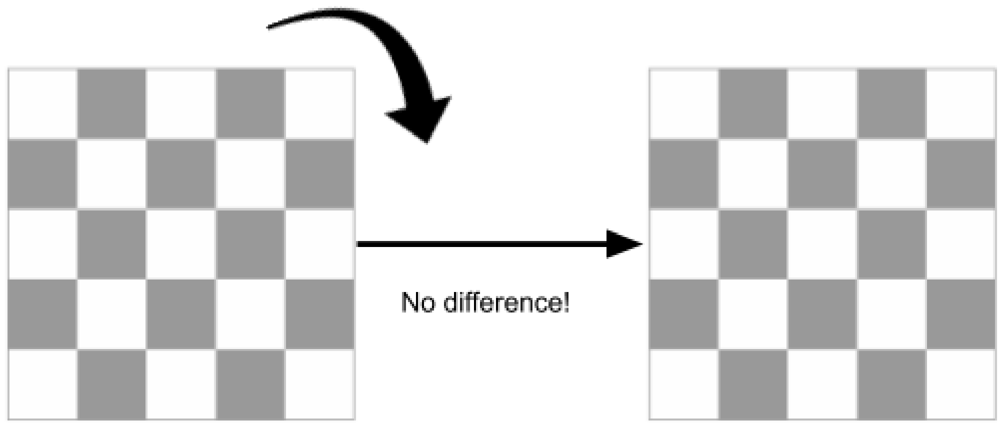
Analysis of cases of m x n boards:

m is bigger than n/m is smaller than n

- We can deduce that when m>n, L is more likely to win
- In a m x n board, the number of safe-making moves L has is equal to $2\lfloor m/2 \rfloor$ if n>3 and the safe-making moves R has is $2\lfloor n/2 \rfloor$ if m>3
- The number of safe-making moves is correlated to the number of safe-moves one will have throughout the whole game
- If the number of safe moves L has is more than R and there are no more safe making moves, L wins
- Thus when m is bigger than n, L has an advantage and is more likely to win, though there are exceptions, such as 2 x 13
- The opposite is true when m is smaller than n, where we can flip the board 90 degrees to switch between the two

m equals n i.e. square board:

- For any square board i.e. equal sides, we know for sure it is either a N or P
- If the board is L or R, when we flip the board 90 degrees, the winner would change from L to R or R to L because R will be playing as L and L will be playing as R
- However, the board remains the same because the four sides are equal and thus when turned 90 degrees, it is still the same block
- Since no board can have two outcomes, we can prove that it is not possible for square boards to be L or R by contradiction


No difference!

6. Conclusion

6.1 Analysis of results

- RQ1:
    - The best strategy is to optimise the difference between the moves you have and your opponent has, in your favour
    - Boards can be split as long as it is not along any optimal move, so we can split it into smaller boards for easier computation
    - Boards can be assigned values for adding them up together, and we can use the summed values to determine the type of board
    - Safe moves are the most important types of moves and the goal of each player is to gain as many as possible before both sides have no safe-making moves
- RQ2:
    - 2 x n and 3 x n boards are favoured towards Right because in 3 x n, Right has multiple double-safe making moves and multiple safe-making moves in 2 x n boards, whilst they are limited for Left
    - All 2 x n boards where $n>27$ and 3 x m boards where $m>4$ are definitely Right wins
- RQ3:
    - Square boards are not biased to either player, since they have the same length
    - All square boards are either first/second player wins
    - m x n boards where $m>n$ tend to be biased towards Left while m x n boards where $m<n$ tend to be biased towards Right since they have more safe-making moves

6.2 Areas for improvement and further research

- RQ3: We could have explored more cases to find more patterns as we only had a few due to time constraints.
- We could also look into irregular-shaped boards (i.e. boards which are not rectangles) as Domineering is not restrained to regular boards or grids.
- Cram is a game with similar rules to Domineering, except the players are not restricted to placing dominoes only horizontally or vertically. (the players just have to align them to the grid squares). Solutions for Cram could be looked into.

7. References

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