When Knights Meet

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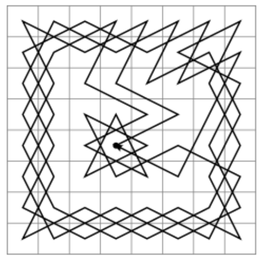
Contents

1. Introduction and Rationale

The Knight's Tour is a common instance of the Hamiltonian Path problem in graph theory. The game is played with a single knight piece on a chessboard where the piece starts from a cell (i,j) and can choose the below move options: [i±1, j±2] or [i±2, j±1]. Moves can vary. Players would need to find a sequence of moves such that the knight has landed on every cell once, forming what is known as a "tour" like this.



This classic problem was first posed almost 1 millennium ago. In *Kavyalankara*, a Sanskrit work on Poetics, the pattern of a knight's tour on a half-board has been presented as an elaborate poetic figure called the *turagapadabandha* or 'arrangement in the steps of a horse'. It was investigated by great mathematician, Leonard Euler, before the first strategy, Warnsdoff's Rule, was discovered in 1823 by H.C. von Warnsdorff.

Due to the extensive history of research done in this topic and the many options to research on in this topic, our group has decided to embark on this project. In this research, we attempt to investigate basic requirements for solutions in research question 1 and a variation to the research questions 2 and 3. In the original problem, one knight is introduced to form a tour. In the variation, we will experiment possibilities of "tours" with the presence of two knights starting at specific cells.

1.1 Objectives

- To discover square boards with the presence of a Knight's tour.
- To find square boards which allow 2 Knights moving simultaneously covering every cell
- Conditions of varying square boards for a Knight's Tour.

1.2 Research Problems

1. In a square board, under what conditions can a Knight's Tour be formed?
2. In a square board, can two knights move simultaneously such that each cell is visited collectively by them exactly once? If yes, how do we form each tour?
3. In a square board, can the two knights form a closed tour each?

1.3 Terminology

Term	Definition
Vertices	Vertices are defined as the endpoint of an edge and the fundamental unit of which graphs are formed.
Edges	Edges are defined as the lines that join a pair of vertices (m,n) together
Cells	A cell is a square in a board.
Graph	A diagram containing vertices and edges that describes the relationship between points.
Closed Knight's Tour	A Closed "tour" is defined as a tour where the knight can end in the same cell which it started from.

1.4 Literature Review

Conrad, A., Hindrichs, T., Morsy, H., Wegener, I. (1994) gave an overview of the conditions required for the presence of a knight's tour. The minimum requirement specified is that on a square n x n board, there would exist an open tour only if $n \geq 5$ whereas $n \geq 6$ and $n=2k$ for a closed tour to form.

Cull, P. and De Curtins, J. (1978) discussed tours formed on rectangular boards in detail. They also set the tours to start from the lower left of the rectangle and found solutions which are able to end at specific varying cells. With the knowledge that there is a Knight's Tour on any large enough m×n board ($m > n \geq 5$), they discovered that for any 5 x m board ($m \geq 5$) there would always exist a tour which can exit on either the lower or upper right of the rectangle. Also, 8 x m board ($m \geq 5$) will always have a closed tour which also can be seen from Conrad's result

Allen J. Schwenk (1991) found that for any m by n board where $m \leq n$, there is always a closed Knight's Tour unless one or more of the conditions are fulfilled:

- m and n are both odd
- m is 1, 2 or 4
- m is 3 and n is 4, 6 or 8

H. C. von Warnsdorff (1823) found that for a Knight on the board, it will proceed to the cell from which there will be the fewest onward moves.

2. Methodology

First, we experimented with different rectangular boards for each of the researched questions either with pen and paper or excel and spreadsheets. Then, we consolidated the patterns and results we observed.

Next, we researched online for relevant articles or works related to our research topic to gain insight and tried to formulate conclusions.

Then, we attempted to prove the results mathematically with formulas and diagrams.

3. Results

We will be using Schwenk's Theorem to help us in the solving of the questions.

Schwenk's Theorem (1991) states that
An $m \times n$ board with $m \leqq n$ will contain a closed Knight's tour if one or more of the following conditions are fulfilled:

- both m and n are odd
- m is 1, 2 or 4
- m is 3 while n is 4, 6 or 8

3.1 Research Question 1

In research question 1, we consider the conditions for a Knight's Tour and to be formed. We found out that for a 1×1, 2×2, 3×3 and a 4×4 square board, there is no Knight's Tour; for a $n \times n$ square board with n at least 5, there is always a Knight's Tour. The full proof can be found in Appendix 7.1.

3.2 Research Question 2

In research question 2, we investigated whether two knights can move simultaneously to cover every cell on the board exactly once when they start from opposite corners. We have reached the following conclusion.

For a $n \times n$ square board with n is odd, there is always no solution.

For a 2×2 and a 4×4 square board, there is always no solution.

For a $2n \times 2n$ board with n at least 3, there is always a solution.

The full proof can be found in Appendix 7.2.

3.3 Research Question 3

In research question 3, we investigate whether two Knights can move simultaneously to cover every cell exactly once while each Knight's tour is closed and they start from different corners.

For a $n \times n$ square board with n is odd, there is always no solution.

For a 2×2 and a 4×4 square board, there is always no solution.

For a $2n \times 2n$ board with n at least 3, there is always a solution.

The full proof can be found in Appendix 7.2.

4. Conclusion

For research question 1, in order to complete a knight's tour, a knight must cover every square of the board. There are two types of knight's tour, an open tour, where the last cell that the knight covered cannot move to the starting cell in one move, and a closed tour, where the last cell that the knight covered can move to the starting cell in one move. We researched the conditions required for a knight's tour to be formed. For 1 x 1 to 4 x 4 boards, there is no knight's tour, whereas for an n x n chessboard, where n is at least 5, there will always be a knight's tour.

For the second research question, we investigated whether 2 knights can simultaneously move such that every cell on the board is covered exactly once when they start from opposite corners. We found that in a n x n board, where n is an odd number, there is no solution, for 2 x 2 and 4 x 4 boards, there is also no solution. For 2n x 2n boards, with n being at least 3, there will always be a solution.

Lastly in research question 3, we researched on whether two knights can move simultaneously to each form a closed knight's tour when they start from different corners and the result is identical to question 2.

5. Extensions

Here are some of the possible areas for further research in the future:

Can there be two knights tour in a n x n square when n is odd and the middle must be empty?

Can two knights tour be formed on a rectangular board?

Can a knights tour be formed on a three dimensional board?

6. References

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Numberphile: Knight's Tour-Youtube https://www.youtube.com/watch?v=ab_dY3dZFHM Retrieved on 20 March 2020

7. Appendix

7.1 Proof of Research Question 1

For a 1×1, 2×2 and a 3×3 square board, it is trivial to see that they do not contain a Knight's tour.

For a 4×4 square board, we use two colouring patterns to prove that there is no Knight's tour:

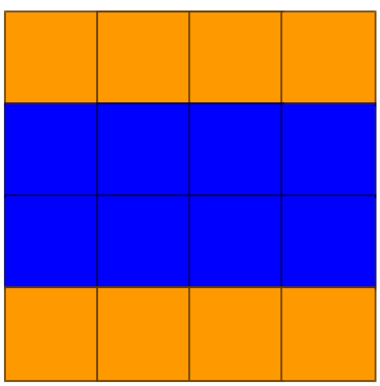

Fig. 1

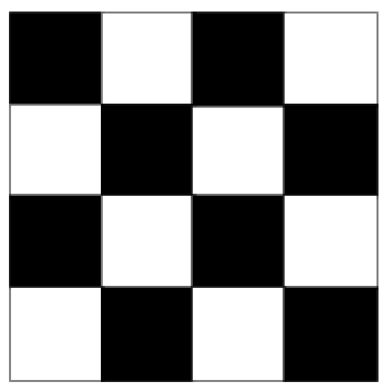

Fig. 2

In Fig.1, note that there is an equal number of orange and blue cells and that a Knight is unable to get to an orange cell from another orange cell. Thus, to visit all the cells, the Knight has to go to a different coloured cell from its last step. Without loss of generality (WLOG), let us assume the Knight starts on an orange cell.

In Fig.2, it is obvious that the Knight has to go in a path with cells with alternating colours in adjacent moves. WLOG, let it start on a black cell.

Now, we find out that the Knight will only reach cells which are orange in Fig. 1 and black in Fig. 2 or blue in Fig. 1 and white in Fig. 2. Those cells which are orange in Fig. 1 and white in Fig. 2 or blue in Fig. 1 and black in Fig. 2 are unable to be reached by the Knight. Thus, there is never a closed tour on a 4 × 4 square board.

For a n × n board with n at least 5, we define two graphs to help with the proving:

-Graph G_n, a graph of the Knight's Tour on a n × n square board C_n with n = 5, 6, 7, 8 or 9.

-Graph L_n, a graph of the Knight's Tour on a Γ-shaped board D_n, with 5 rows of n cells and (n-5) rows of 5 cells in order from top to bottom, with n = 6, 7, 8 or 9.

Here are the graphs for G_5, G_6, G_7, G_8, G_9, L_6, L_7, L_8 and L_9:

G_5

1	18	13	22	7
12	23	8	19	14
17	2	21	6	9
24	11	4	15	20
3	16	25	10	5

G_6

1	22	19	8	33	28
18	7	34	29	20	9
23	2	21	32	27	30
6	17	4	35	10	13
3	24	15	12	31	26
16	5	36	25	14	11

G_7

1	4	23	40	11	6	9
24	41	2	5	8	45	12
3	22	47	44	39	10	7
42	25	34	21	46	13	38
31	28	43	48	15	20	17
26	35	30	33	18	37	14
29	32	27	36	49	16	19

G_5

G_5

1	34	3	18	37	32	13	16
4	19	36	33	14	17	38	31
35	2	49	52	47	40	15	12
20	5	64	41	50	53	30	39
57	42	51	48	59	46	11	26
6	21	58	63	54	27	60	29
43	56	23	8	45	62	25	10
22	7	44	55	24	9	28	61

1	36	3	22	49	38	65	20	17
4	23	48	37	78	21	18	39	66
35	2	53	50	47	64	77	16	19
24	5	34	81	52	79	46	67	40
33	58	51	54	69	74	63	76	15
6	25	70	59	80	45	68	41	62
57	32	55	28	73	60	75	14	11
26	7	30	71	44	9	12	61	42
31	56	27	8	29	72	43	10	13

L_6

32	3	14	9	26	1
13	24	33	2	15	10
4	31	12	25	8	27
23	20	29	34	11	16
30	5	18	21	28	7
19	22	35	6	17	

L_7

9	6	11	34	23	4	1
12	31	8	5	2	29	24
7	10	33	30	35	22	3
32	13	40	21	28	25	36
17	20	15	44	37	42	27
14	39	18	41	26		
19	16	45	38	43		

L_8

46	23	6	25	30	35	4	1
7	26	45	36	5	2	31	34
22	47	24	29	52	33	42	3
27	8	49	44	37	40	51	32
48	21	28	53	50	43	38	41
9	12	19	16	39			
20	17	14	11	54			
13	10	55	18	15			

L_9

27	42	37	30	21	14	35	4	1
36	31	28	43	38	7	2	31	34
41	26	33	22	29	20	33	42	3
32	35	24	39	44	3	40	51	32
25	40	45	34	23	16	43	38	41
46	63	50	57	54				
51	58	55	62	49				
64	47	60	53	56				
59	52	65	48	61				

For a 5 × 5, 6 × 6, 7 × 7, 8 × 8 and a 9 × 9 square board, there is a solution as shown in the graphs G_5, G_6, G_7, G_8 and G_9.

When the board has a side length n larger than 9 and n is a multiple of 5, let k be the remainder when n is divided by 5. Now, regardless whether n is a multiple of 5, we can cut out a (n-5) × (n-5) square board in the bottom right corner. Then, cut out a board D_{k+1} on the top left corner. After that we will be able to split the remaining spaces into C_5.

Starting from the top right corner, the Knight travels through the C_5s in the top row, using the tour show in G_5 but rotated 90° clockwise, so that the end of a G_5 in a C_5 can jump to the start of the tour of the next C_5 in 1 move. Then, it will complete L_{k+1} then go down the column of C_5s, ending up in the bottom left corner of the remaining (n-5) × (n-5) square board. A similar process repeats until there is a C_5, C_6, C_7, C_8 or a C_9 left, where a tour exists.

Thus, there is no tour in a n × n square board when n is 1, 2, 3 or 4, while other cases all hold a solution.

7.1 Proof of Research Questions 2 and 3

Since research questions 3.2 and 3.3 have similar results, we will prove them together.

For a n × n square board, where n is 2 or n is odd, it is easy to see that there is no tour at all.

When n = 4, using a similar colouring shown in Fig.1 and Fig. 2, the two Knights start on the same colour for both figures as they start at opposite corners. WLOG let it be orange in Fig. 1 and black in Fig. 2. With a similar argument as Appendix 7.1, the cells orange in Fig.1 and white in Fig. 2 or blue in Fig. 1 and black in Fig. 2 cannot be reached. Thus there is no tour on a board with n = 4.

When n = 6 or n = 8, here are the solutions:

1	10	13	4	17	8
14	5	18	9	12	3
11	2	15	6	7	16
16	7	6	15	2	11
3	12	9	18	5	14
8	17	4	13	10	1

1	8	21	14	29	6	19	12
22	15	32	7	20	13	28	5
9	2	25	30	27	4	11	18
16	23	10	31	24	17	26	3
3	26	17	24	31	10	23	16
18	11	4	27	30	25	2	9
5	28	13	20	7	32	15	22
12	19	6	29	14	21	8	1

Since the tours shown are closed for 2 Knights, there is both a solution for research question 3.2 and 3.3 when n is 6 or 8.

By Schwenk's Theorem, an n × 2n board with n at least 5 always has a closed tour. Thus, for a 2n × 2n board, if we cut the board into two n × 2n boards, there is a solution if each Knight does a tour in one of the rectangle boards.

Thus, there is a solution for a n × n board for both research questions 3.2 and 3.3 if n is not odd and not 2 or 4.