Group 8-02
The 24 Game

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1 Introduction

1.1 Aim And History of our Project

The modern version of the 24 Game was invented and copyrighted by engineer and inventor Robert Sun in 1988. The first edition of the game, Single Digits, started in schools in 1989. The 24 Game is copyrighted, and is produced by Pennsylvania-based Suntex International Inc. Through this project , we would like to promote the 24 Game to more people (since this is not a popular game) and use it as the theme of our project.

1.2 Gameplay

1.2.1 Poker Cards and Valid Equations

The only item used in this project is a deck of poker cards. (Refer to Figure 1) All 52 cards will be used in the game, excluding Jokers. In the game , we will use 'Ace' as 1 , '2' to '10' by face value , 'J' as 11 , 'Q' as 12 and 'K' as 13. (Refer to Figure 2)

Figure 1 - A deck of poker cards

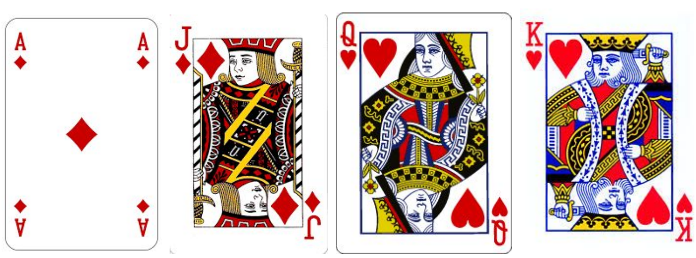

Figure 2 - The set of cards that are used as 1 , 11 , 12 and 13 respectively

The main term used in this project is ‘a valid equation’. A ‘valid equation’ refers to an equation that allows you to remain in the game. For an equation to be deemed as a ‘valid’ one, here are the 5 rules :

- The operations used must only be addition , subtraction , multiplication , division and/or brackets (Refer to Figure 3)
- 24 must be the result of the equation formed (Refer to Figure 4)
- Cards are not allowed to be combined with other cards (e.g. 2 and 3 into 23)
  (Refer to Figure 5)
- Cards are not allowed to be flipped over (e.g. 6 to 9 and vice versa) (Refer to Figure 6)
- All cards must be used (Refer to Figure 7)

Do note that the position of cards may be switched.
For example , {2,2,4,8} can be switched to {2,4,8,2}.

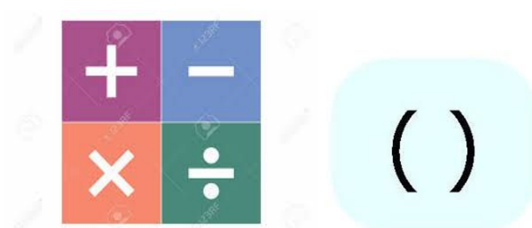

Figure 3 - The main operations that are allowed in valid equations

$$= 24$$

Figure 4 - The result of the equation must be 24

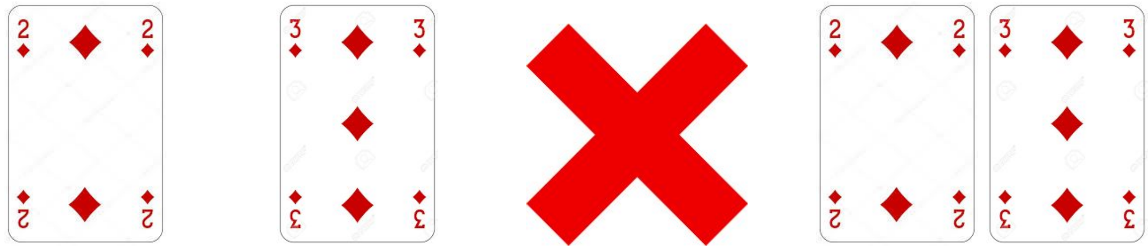

Figure 5 - Cards cannot be combined with other cards

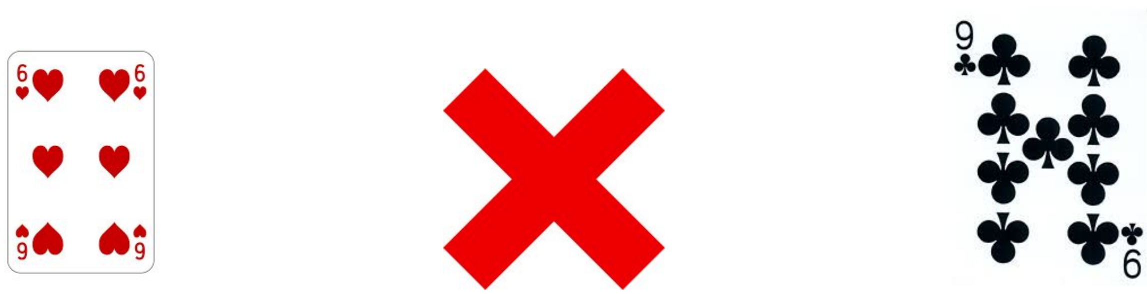

Figure 6 - Cards cannot be flipped over

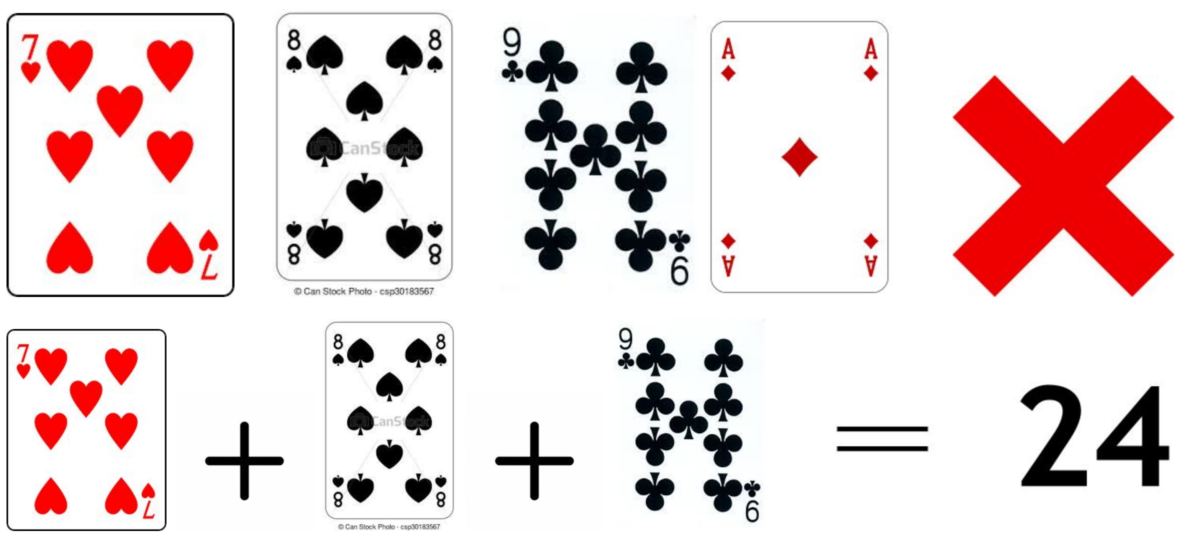

Figure 7 - All cards must be used

1.2.2 Starting The Game

First, players take the deck of 52 cards and remove all Jokers. After that, the players place the deck of cards in front of them, and sit around the deck of cards. The first player starts the game by picking 4 cards from the deck. (Refer to Figure 8)

Figure 8 - An example of how to start the game (for 4 players)

1.2.3 Making A Move

A player has to try to form a 'valid equation' using his/her cards that he/she drew. Once a 'valid equation' is formed, the 4 cards would go back into the deck and the deck of cards will be re-shuffled. (Refer to Figure 9)

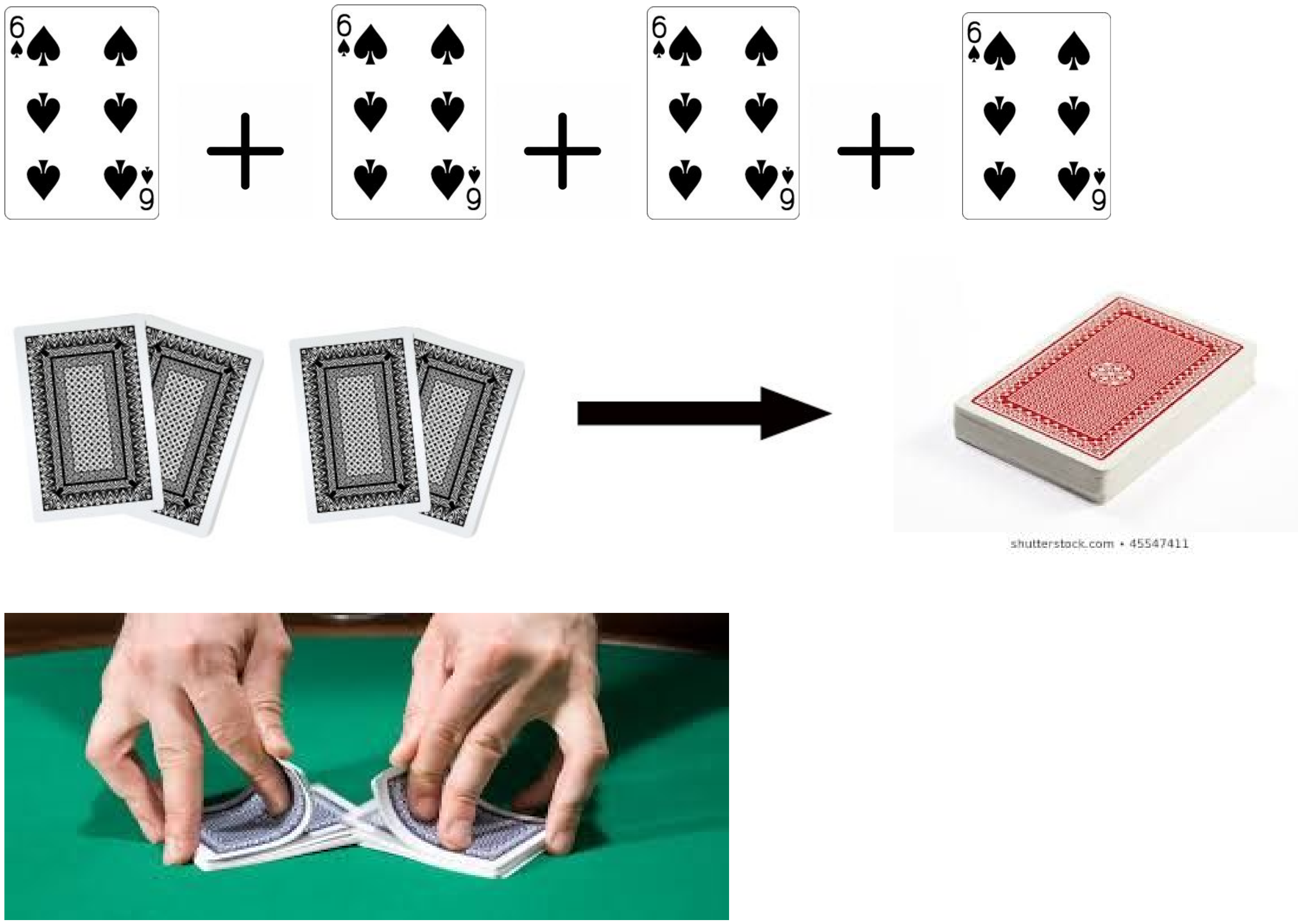

Figure 9 - Obtaining a valid equation , cards going back into the deck and re-shuffling of all the cards

However , if a ‘valid equation’ cannot be formed with the cards picked , the 4 cards would still go back into the deck and be re-shuffled, but the player would be disqualified from the game. Hence, the player leaves the game and loses immediately. (Refer to Figure 10)

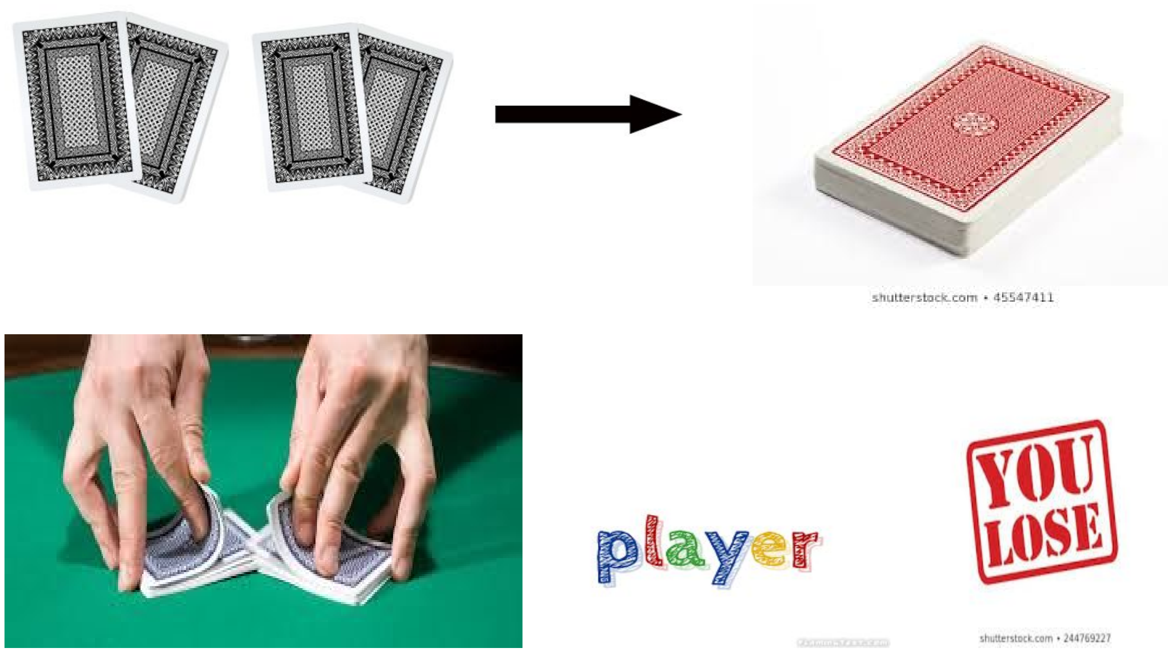

Figure 10 - Cards still go back into the deck and re-shuffled , but the player loses

1.2.4 Ending The Game

The game ends when among n players, (n-1) players have been disqualified , either by not following the rules above , or have taken cards which are not able to form a 'valid equation' no matter what operation signs were used. The winner of the game is the last player that has not been disqualified. All the other players , regardless of when they have been disqualified , are considered losers in the game.

1.3 Literature Review

https://www.24game.com/t-teachers-ACT13.aspx

This webpage is written by a former math teacher, on strategies to win the 24-game. Although this game is not the original draw card style, it still has the same concept of the theme of this project. Sometimes the game requires a tinge of luck, but this teacher shows us some math techniques that can be used throughout the game. For example, she teaches us about making groups and making ones.

https://www.youtube.com/watch?v=HeO0q0cSvx0
This is a simple tutorial by a girl on how to play the 24 game using the real poker cards.

2. Research questions

Our research questions are as follows :
RQ1: What is the probability that 4 random cards picked from the range of 1 to 10 can form a valid equation, provided the numbers don't repeat?
RQ2: What is the probability that 4 random cards picked from the range of 1 to 10 can form a valid equation? However, this time repetition of the numbers is allowed.
RQ3: What is the probability that 4 random cards picked from the range of 1 to 13 can form a valid equation? Repetition of the numbers is allowed.

Research Questions 1, 2 and 3 will be answered in Sections 3 , 4 and 5 respectively.

3. RQ1 - What is the probability that 4 random cards picked from the range of 1 to 10 can form a valid equation, provided the numbers don't repeat?

3.1 Information

Since we are only talking about numbers ranging from 1 to 10 , we are removing the cards J , Q and K. As only distinct numbers are used in each set, we can also decrease the number of cases that need to be checked by us.

Since we have 10 numbers {1,2 ... 10} and all numbers have to be distinct, we can easily calculate the total number of cases that needs to be checked.
Total number of cases : 10C4 = 210
In this case , since the smallest 4 cards we can get is {1,2,3,4} and there is a solution being 1*2*3*4 = 24 , we have to check all 210 cases.

3.2 Solution & Conclusion

We have thoroughly checked all the cases with 13 of them not producing a ‘valid equation’. We will provide all of these 13 cases and 13 extra ‘valid cases’. (Refer to Figure 11)

No.	Not Valid	Valid	Valid (Operation)
1	{1,2,9,10}	{1,2,3,4}	1×2×3×4=24
2	{1,3,4,6}	{4,5,6,9}	4+5+6+9=24
3	{1,4,7,10}	{3,4,5,6}	6(3+5-4)=24
4	{1,4,8,10}	{1,2,5,10}	2×10+5-1=24
5	{1,6,7,8}	{2,3,4,5}	2(3+4+5)=24
6	{1,8,9,10}	{5,6,7,8}	(8-6)(5+7)=24
7	{2,7,8,10}	{1,5,7,9}	(9-5)(7-1)=24
8	{3,4,9,10}	{7,8,9,10}	9×8÷(10-7)=24
9	{3,5,8,10}	{2,3,5,6}	6×5-2×3=24
10	{3,7,8,10}	{2,4,8,10}	4×10-8×2=24
11	{5,6,7,10}	{1,7,8,10}	8+10+7-1=24
12	{5,8,9,10}	{1,5,9,10}	1(5+9+10)=24
13	{6,7,9,10}	{2,3,4,10}	3×10-2-4=24

Figure 10 - The table of 13 invalid and valid cases

Out of the 210 cases, a total of 197 cases provide valid equations. Thus, the probability of getting a valid equation (so that you are not guaranteed to lose) is **93.8%** (3 sf).

4 RQ2 - What is the probability that 4 random cards picked from the range of 1 to 10 can form a valid equation? However, this time the numbers can repeat.

4.1 Information

There are complications in this Research Question , which also applies for the third (or the last) Research Question.

This is because the numbers are allowed to be repeated. Since we are using Poker Cards , there will be complications. These complications will be covered in Sections 4.1.1 and 4.1.2 respectively.

4.1.1 Complication 1 - Probability of Getting A Certain Set of Cards

The first complication - The probability of getting a certain set of cards. This means that to get a certain set of cards {a,b,c,d} , there are different chances. This is actually different from the first Research Question because the chance of getting for example , {1,2,3,4} or {5,6,7,8} , is the same. However , in the second Research Question , the chance of getting for example , {5,6,6,7} and {6,6,6,6} is different because getting {6,6,6,6} is rarer than getting {5,6,6,7}. This needs to be considered in our solving of the Research Question.

Hence , we have come up with a plan. Let different letters represent distinct positive integers from 1 to 10. There are 5 possible sets. They are {a,b,c,d} , {a,a,b,c} , {a,a,b,b} , {a,a,a,b} and {a,a,a,a}. All the sets in each of the above (e.g. {a,a,b,b}) have the same chance. For example , {5,5,6,6} and {3,3,9,9} have the same chance of being picked. This will be reflected in our simple table. (Refer to Figure 12) Note that this is not the actual table for our project.

	Number of cases that work	Total number of cases
All numbers are distinct		
3 numbers are distinct		
There are 2 pairs of numbers		
3 of a number and 1 of another number		
All the numbers are the same		

Figure 12 - A simple table that reflects the plan for Complication 1

This would be useful for us in solving this problem.

4.1.2 Complication 2 - The Type/Suit of Cards

The second complication - The type or the suit of the cards used. This is because since the cards can be repeated , we must ensure that a set has repeated numbers , the type or suit of the cards needs to be considered as for example for {5,6,6,7} , the 2 ‘6’s cannot be both hearts or clubs. Hence , we will introduce another modification to the table - in order to include the change in ‘suit’ or ‘type’ of cards. However , in this case (to avoid complication or confusion) , in {5,6,6,7} , the ‘5’ and ‘7’ will also have 4 suits available too.

This modification will include the introduction of more terms. The table and the terms used (and their definitions) will be reflected in the table below. (Refer to Figure 13)

	Number of ways to modify the suits to make distinct combinations	Number of methods getting a certain combination of this type in a deck of cards (suit matters)	Number of sets* which don't work	Number of cases^ which don't work
All 4 numbers are distinct				
3 numbers are distinct				
There are 2 pairs of numbers				
3 of a number and 1 of another number				
All 4 numbers are the same				

* The number of sets refers to the number of possible {a,b,c,d} (where a , b , c and d need not be distinct) that can be formed

^ The number of cases refers to the number of sets , taking into account the suits of the Poker Cards.

Figure 13 - The table with the modification for Complication 2

4.1.3 Final Merge of Complications 1 & 2 - Table

The table below will show the table we are going to use for our Project Work in **Research Question 2** - taking into account both complications we have raised above. (Refer to Figure 14 on the next page)

Figure 14 - Final table taking into account both problems we raised above	Number of distinct sets of 4 numbers that satisfy the conditions	Given a set , how many ways are there to modify the suits to make distinct combinations?	Number of methods getting a certain combination of this type in a deck of cards (suit matters) = Number of distinct sets multiplied by the number of ways there is to modify the suits	Number of sets which don't work	Number of cases which don't work = Number of sets which don't work multiplied by the ways to modify the suits	Number of cases that work = Number of methods of getting a combination minus the number of cases which don't work
All 4 numbers are distinct						
3 numbers are distinct						
There are 2 pairs of numbers						
3 of a number and 1 of another number						
All 4 numbers are the same						

4.2 All Numbers Are Distinct

This section is very similar to RQ1 , just that it involves the second complication - the suit* of the cards used.
*For reference , the suit refers to the type of cards (e.g. hearts).

The total number of sets is 210. (This was brought up in RQ1).
There are a total of $4 \times 4 \times 4 \times 4 = 256$ ways to modify the cards.
This is because for each distinct number in {a,b,c,d} , there are 4 suits it can take (spades , hearts , diamonds and clubs).

The total number of cases if all numbers are distinct = $210 \times 256 = 53760$.

In RQ1 , since we have found that there were 13 cases which don't work , we can also say that the number of sets which does not work is 13.

Hence , the number of cases which doesn't work is $13 \times 256 = 3328$.

This leaves us with $53760 - 3328 = 50432$ 'valid cases'.

All this information can be presented in a table. (Refer to Figure 15)

	Number of distinct sets of 4 numbers that satisfy the conditions	Given a set , how many ways are there to modify the suits to make distinct combinations?	Number of methods getting a certain combination of this type in a deck of cards (suit matters)	No. of sets which don't work	No. of cases which don't work	Number of cases that work
All 4 numbers are distinct	210	256	53 760	13	3328	50 432

Figure 15 - Table of information (All Numbers Are Distinct)

4.3 3 Numbers Are Distinct

This section implies that there is a number which repeats itself , hence the format is {a,a,b,c}. It also involves the suit used.

The total number of possible sets is 360 , which can be derived in 2 ways.
Way 1. Choose ‘a’ first. There are 10 choices. Next , choose ‘b’ and ‘c’ , which has 9C2 = 36 ways. $10 \times 36 = 360$
Way 2. Choose ‘b’ and ‘c’ first. There are 10C2 = 45 choices. Next , choose ‘a’ , which has 8 ways. $45 \times 8 = 360$
There are a total of $4 \times 4 \times$ 4C2 = 96 ways to modify the cards.
This is because for ‘a’ , there are 4C2 ways and for ‘b’ and ‘c’ , there are 4 ways each.

Total number of cases = $360 \times 96 = 34560$

We have found 66 invalid sets , and with the suits , there are 6336 cases.

This leaves us with 34560 - 6336 = 28224 ‘valid cases’.

All this information can be presented in a table. (Refer to Figure 16)

	Number of distinct sets of 4 numbers that satisfy the conditions	Given a set , how many ways are there to modify the suits to make distinct combinations?	Number of methods getting a certain combination of this type in a deck of cards (suit matters)	No. of sets which don’t work	No. of cases which don’t work	Number of cases that work
3 numbers are distinct	360	96	34 560	66	6336	28 224

Figure 16 - Table of Information (3 Numbers Are Distinct)

4.4 There Are 2 Pairs of Numbers

This section implies that the format of numbers is {a,a,b,b}.

The total number of possible sets is 45 , because there are 10C2 = 45.
You choose the 2 numbers , ‘a’ and ‘b’.
There are a total of 4C2 × 4C2 = 36 ways to modify the cards (suit).

Total number of cases = 36 × 45 = 1620

We have found 19 invalid sets , and with the suits , there are 684 cases.

This leaves us with 1620 - 684 = 936 ‘valid cases’.

All this information can be presented in a table. (Refer to Figure 17)

	Number of distinct sets of 4 numbers that satisfy the conditions	Given a set , how many ways are there to modify the suits to make distinct combinations?	Number of methods getting a certain combination of this type in a deck of cards (suit matters)	No. of sets which don’t work	No. of cases which don’t work	Number of cases that work
There are 2 pairs of numbers	45	36	1 620	19	684	936

Figure 17 - Table of Information (There Are 2 Pairs of Numbers)

4.5 3 of A Number and 1 of Another Number

This section implies that the numbers are in the format {a,a,a,b}.

There are 10 × 9 = 90 possible sets.
There are a total of 4C3 × 4 = 16 ways to modify the cards (suit).

Total number of cases = 90 × 16 = 1440

We have found out that there are a total of 41 invalid sets.

Considering their suits , we have 41 × 16 = 656 invalid cases.

This leaves us with 1440 - 656 = 784 'valid cases'.

All this information can be presented in a table. (Refer to Figure 18)

	Number of distinct sets of 4 numbers that satisfy the conditions	Given a set , how many ways are there to modify the suits to make distinct combinations?	Number of methods getting a certain combination of this type in a deck of cards (suit matters)	No. of sets which don't work	No. of cases which don't work	Number of cases that work
3 of a number and 1 of another number	90	16	1 440	41	656	784

Figure 18 - Table of Information (3 of A Number and 1 of Another Number)

4.6 All 4 Numbers Are The Same

This section is simple - the cases are in the format {a,a,a,a}.

There are a total of 10 possible sets : {1,1,1,1} to {10,10,10,10}.
There's only one way to modify the cards used (suit).

Total number of cases = $10 \times 1 = 10$

We have found 6 sets which don't work , which is equal to 6 cases which don't work.

Since the number of cases is small , we list the 10-6=4 'valid cases'.

{3,3,3,3} ---- $3\times3\times3-3 = 24$
{4,4,4,4} ---- $4\times4+4+4 = 24$
{5,5,5,5} ---- $5\times5-(5\div5) = 24$
{6,6,6,6} ---- $6+6+6+6 = 24$

All this information can be presented in a table. (Refer to Figure 19)

	Number of distinct sets of 4 numbers that satisfy the conditions	Given a set , how many ways are there to modify the suits to make distinct combinations?	Number of methods getting a certain combination of this type in a deck of cards (suit matters)	No. of sets which don't work	No. of cases which don't work	Number of cases that work
All numbers are the same	10	1	10	6	6	4

Figure 19 - Table of Information (All Numbers Are The Same)

4.7 Solution & Conclusion

4.7.1 Brief Idea of What Has Been Done

By breaking down into the 5 scenarios shown above , we have explained how we derived our answers and solutions for each row and each column in the table that we have shown in a table above. (Refer back to Figure 14)

4.7.2 Intention to Present Our Solutions

We intend to present our solutions in the table we have shown. (Refer to Figure 14) We will fill up the table with the necessary information we have deduced and explained in the earlier sections from 4.2 to 4.6. Hence , we are going to do a final combination of all the information and present them clearly. (Refer to Figure 20)

4.7.3 Inferences & Conclusions We Have Made

We have inferred the following :

- When solving these kinds of problems with a certain/special type of card , we need to take into account the suit of cards (in Poker).
- When solving problems with repetition of numbers , the chance of being picked is also very crucial and important as it affects the solution. (We are talking about probability here)
- As we go from Sections 4.2 to 4.6 , the chance of having a valid equation decreases. In Section 4.2 , the chance of being picked is 50432÷53760 = 0.938 (to 3 s.f.) and in Section 4.6 , the chance of being picked is 4÷10 = 0.4.
- We are restricting the number range of cards (1 to 10) in this Research Question and it already has about 90000 cases. If we look at the real scenario (RQ3) , the number of cases would surely increase by a whole lot.

Figure 20 - Final Solutions (RQ2)	Number of distinct sets of 4 numbers that satisfy the conditions	Given a set , how many ways are there to modify the suits to make distinct combinations?	Number of methods getting a certain combination of this type in a deck of cards (suit matters) = Number of distinct sets multiplied by the number of ways there is to modify the suits	Number of sets which don’t work	Number of cases which don’t work = Number of sets which don’t work multiplied by the ways to modify the suits	Number of cases that work = Third grid minus the fifth grid
All 4 numbers are distinct	210	256	53 760	13	3 328	50 432
3 numbers are distinct	360	96	34 560	66	6 336	28 224
There are 2 pairs of numbers	45	36	1 620	19	684	936
3 of a number and 1 of another number	90	16	1 440	41	656	784
All 4 numbers are the same	10	1	10	6	6	4

4.7.4 Final Solution To The Research Question

Recall that although we have found all the necessary information, the answer to the Research Question has not been calculated yet. The solution is as follows :

Total Number of Cases = 53 760 + 34 560 + 1 620 + 1 440 + 10 = 91 390
Cases That Work = 50 432 + 28 224 + 936 + 784 + 4 = 80 380

Thus, the probability of getting a valid equation (so that you are not guaranteed to lose) is **88.0%** (3 sf).

5 RQ3 - What is the probability that 4 random cards picked from the range of 1 to 13 can form a valid equation? The numbers can repeat.

5.1 Information

The complications here are exactly the same as those in Research Question 2. We are just going to briefly describe what the complications are and how we intend to solve it. After viewing the complications , note that we will still use the same table for this Research Question. (Refer back to Figure 14)

5.1.1 Complications - Probability of Getting A Certain Set of Cards & The Type/Suit of Cards

The probability of getting a certain set of cards matters because apparently , the cards are allowed to repeat. So the chances are different to pick a card as {5,6,6,7} and {6,6,6,6} has a different probability of being picked. This will be taken into account.

The type/suit of cards also matters (spades , hearts , diamonds and clubs) because we are using Poker Cards. Hence in {5,6,6,7} , the 2 '6's must be of different suits.

Hence we will use the table to solve this question too. This time, there are 270,725 different cases.

5.2 All 4 numbers are distinct

As mentioned earlier there are 256 ways to modify a set of 4 distinct numbers into suits.
This time, 102 out of the 715 sets of numbers do not work, thus 156,928 cases ((715-102)(256)) will work.

5.3 3 Numbers are Distinct

As mentioned earlier there are 96 ways to modify a set of 3 distinct numbers (1 repeated number) into suits.
This time, 244 out of the 858 sets of numbers do not work, thus 58,944 cases ((858-244)(96)) will work.

5.4 There are 2 pairs of numbers

As mentioned earlier there are 36 ways to modify 2 pairs of numbers into suits.
This time, 34 out of the 78 sets of numbers do not work, thus 1,584 cases ((78-34)(36)) will work.

5.5 3 of a number and 1 of another number

As mentioned earlier there are 16 ways to modify 3 of a number and 1 of another number into suits.
This time, 70 out of the 156 sets of numbers do not work, thus 1,376 cases ((156-70)(16)) will work.

5.6 All 4 numbers are the same

As mentioned earlier there is 1 way to modify a set of 4 same numbers into suits.
This time, 8 out of the 13 sets of numbers do not work, thus 5 cases ((13-8)(1)) will work.

5.7 Solution & Conclusion

Hence , using the table below (refer to Figure 21) , we have concluded that in the real 24 Game , the probability of getting a valid equation (so that you are not guaranteed to lose) is **80.8%** (to 3 s.f.)

Total Number of Cases = 183040 + 82368 + 2808 + 2496 + 13 = 270725
Cases That Work = 156928 + 58944 + 1584 + 1376 + 5 = 218837

Figure 21 - Final Solutions (RQ3)	Number of distinct sets of 4 numbers that satisfy the conditions	Given a set , how many ways are there to modify the suits to make distinct combinations?	Number of methods getting a certain combination of this type in a deck of cards (suit matters) = Number of distinct sets multiplied by the number of ways there is to modify the suits	Number of sets which don't work	Number of cases which don't work = Number of sets which don't work multiplied by the ways to modify the suits	Number of cases that work = Third grid minus the fifth grid
All 4 numbers are distinct	715	256	183 040	102	26 112	156 928
3 numbers are distinct	858	96	82 368	244	23 424	58 944
There are 2 pairs of numbers	78	36	2 808	34	1 224	1 584
3 of a number and 1 of another number	156	16	2 496	70	1 120	1 376
All 4 numbers are the same	13	1	13	8	8	5

6 Final Overall Conclusion

6.1 Research Questions and Their Solutions

Once again , as aforementioned , our 3 research questions in order are as follows :

- What is the probability that 4 random cards picked from the range of 1 to 10 can form a valid equation, provided the numbers don't repeat?
- What is the probability that 4 random cards picked from the range of 1 to 10 can form a valid equation? However, this time the numbers can repeat.
- What is the probability that 4 random cards picked from the range of 1 to 13 can form a valid equation? The numbers can repeat.

For our first question , **93.8% (to 3 s.f.)** was eventually the final answer , which was derived through the checking of the 210 cases using Mathematics.

For our second question , **88.0% (to 3 s.f.)** was eventually the final answer, which was derived through finding the methods to solve the 2 problems that came out when the numbers are allowed to repeat.

For our third question (real scenario) , **80.8% (to 3 s.f.)** was eventually the final answer , which was derived through using the methods in RQ2 and trying out the cases.

6.2 Conclusions Made

When repetition of numbers is allowed , the chance of getting a 'valid equation' decreases.

When numbers are in a bigger range , the chance of getting a 'valid equation' also decreases.

When numbers are allowed to repeat , the suit of cards and their chance of being picked also matters.

----- THE END -----