Square Manoeuvre

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Contents

1. Introduction

We will consider the following question:

In a classroom, the desks are arranged in 5 rows, 5 desks per row. A student occupies each desk. Each student is asked to move into a neighbouring seat-forward, backward, left or right. They are not allowed to remain in their original seat. They are not allowed to move diagonally. Is it possible to do it such that no two students share the same desk?

1.1 Objectives

The objectives are as follows:

1. To determine the possibility of the question
2. To consider variations of the questions and determine their solutions

1.1 Research Questions

The research questions are as follows:

1. Why is the question unsolvable?
2. How does the number of steps taken by each person affect the results?
3. How does allowing each person to move diagonally affect the results?
4. How does the shape of the grid affect the results?

1.3 Terminology

Term	Definition
Cell	Each grid square being moved to another grid square
Steps	Every movement from one grid square to a grid square that it was not originally occupy
Grid shape	The outline of the entire grid
Unit	The length of the entire grid compared to the length of its constituent grid shapes
Vertex	The point at which two or more lines intersect one another
Diagonally	The grid square in which one of the vertex is shared with another grid square
Adjacent	The grid square in which one of the faces is touching another grid square
Arbitrary Graphs	Graphs that are constructed randomly to be inclusive of all possible scenarios
Directed Graphs	Graphs that are made up of vertices and connected by straight lines or curved lines, arcs

1.4 Literature Review

In 1996, Ravi Vakil, published “A Mathematical Mosaic: Patterns & Problem Solving”. He briefly mentions the background of Eugenia Malinnikova, a mathematician whose insight was inspired by this question. In the question, she solves it with help of a professor who coloured the question into a checkerboard and was proved to be impossible to solve. Such colouring and invariants will be useful in our problem and recreating the results found. However, in our variations, more steps need to be taken to prove that the grid has a certain number of cells to fulfill the requirement needed to successfully prove with checkerboard colouring.

In 1997, Ross Honsberger, a professor, published “In Polya’s Footsteps: Miscellaneous Problems and Essays”. It briefly focuses on the original problem and proofs to show that it was impossible to solve. He then goes on to discuss variations of the problem by changing its dimensions to different quadrilaterals. He discusses smaller cases where one dimension is fixed at 2 and discusses 4 possible outcomes and solve each outcome. A paper from Robert E. Kennedy and Curtis Cooper also talk about the variations on dimensions with one dimension fixed at 3. This is useful in our variations of grid shape which would be using this proof as well as others for different shapes.

In 2013, Daryl Deford published a research paper titled “Counting Combinatorial Rearrangement, Tiling with Squares, and Symmetric Tilings.” It discusses the questions and reconstructs the question into a graph. He uses graphs to reconstruct the questions and redefines the rules in terms of the graphs. He then uses directed graphs to show the movement of each vertex. This can be useful in the problem to reconstruct for the second variations of moving diagonally as it would simplify the questions.

2. Methodology

Firstly, we conducted research on the techniques and ideas potentially applicable to our project, such as graph theory. Next, we use those techniques to recreate solutions to each of the research questions. Finally, we will devise the solutions to each of our questions.

3. Results

3.1 Research Question 1

By using checkerboard colouring, we can find out that without loss of generality, there are 13 black tiles and 12 white tiles. By the pigeonhole principle, one of the white tiles will contain $\lceil 13/12 \rceil = 2$ black tiles. This is not possible as no cells can contain two neighbouring cells after the directive has been carried out. Hence, using this proof, we can recreate the results and say that the question is unsolvable.

3.2 Research Question 2

We can split the question into 2 cases.

One case is when the total number of cells are odd and the other is when the toal number of cells are even.

In Case 1, it is not possible to do so.

By binding each cell with another cell that is n steps away, we can form ⌈ total number of cells/2⌉ of binds leaving an odd number of cells behind as $2n+1-2x \cong 1 \pmod 2$

In Case 2, we can label each cell as the initial coordinates(x_1,y_1) and the ending coordinates(x_2,y_2).

Without loss of generality,

If $x_2>x_1$ and $y_2>y_1$, it would mean that the cell moved upwards and forwards.

If $x_2>x_1$ and $y_2<y_1$, it would mean that the cell moved downwards and forwards.

If $x_2=x_1$ and $y_2>y_1$, it would mean that the cell moved upwards.

By using lines of symmetry to divide the grid square, we can reflect the path line of the cell and mirror the movements such that the line does not divide any cell.

Hence, it is always possible for the second case.

3.3 Research Question 3

In a square, the length would equal to the breadth.

To determine the possibility of the question, we would have to calculate the possible number of steps that can be taken and ensure that it is divisible by 2.

Let $n=l=b$.

The calculation would be $(n-1)^2\binom{4}{2}-(n-1)(n-2)-(n-1)(n-2)=6(n^2-2n+1)-2n^2+6n-4=6n^2-12n-2n^2+6n+2=4n^2-6n+2$

Since $4n^2 \fallingdotseq 6n \fallingdotseq 2 \fallingdotseq 0 \pmod 2$

Any addition equation inclusive of them solely would result in an even number.

Hence, this variation is possible.

3.4 Research Question 4

Triangles

All polygons are enclosed figures. An enclosed figure is made up of lines of which two of them are joined to form an edge of the figure.

In all n-sided polygon, (n-2) triangles can be formed. If the total number of cells in the grid are odd numbered then it would be impossible to carry out the directive.

However, if the total number of cells in the grid are even numbered, it would be possible with the exception of the grid shape being a triangle. This is because they are an edge can only remove two sides from being adjacent to other cells. If the figure is a triangle, it can only be made up of base triangles and not other polygons. This would result in there being one side adjacent to other cells. This would force the exchange between them. However, triangles along the edge are always more than the triangles adjacent to them, hence if the grid shape outline is a triangle, it is not possible to solve. However, if the grid shape is other polygon, taking away two sides would result in a leftover of two or more adjacent sides to other cells. This would increase the total number of steps that are possible and using question 2, we can prove that this is possible. Hence if the total number of cells in the grid are even and the grid shape is that of another polygon other than a triangle, it is possible.

4. Extensions

In conclusion, the directory is only possible when:

The total number of cells in the grid are even

Diagonal steps are allowed between cells sharing the same vertex
The grid shape is that of other polygons that are not triangular when the grid cells are triangular
Possible areas for improvement are:

1. Not limiting the shapes to shapes that can tessellate and regular polygons
2. Extending the solutions to a three dimensional problem like a cube
3. Changing the dimensions of the grid and not the shape

5. References

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