Braids and Knots

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1 Introduction

1.1 Description

Our project is about a classic riddle, to hang a painting on two pins such that removal of any of the pins results in the painting falling. This puzzle can easily be modified, and it falls under the field of mathematics called Braid Theory.

1.2 Rationale, Objectives and Research Questions

1.2.1 Our Rationale

Our rationale was that this puzzle was fairly simple, yet intriguing at the same time. Also, the puzzle could be easily varied to increase its complexity and it could also lead to more knowledge about knots in mathematics, a subject we were all interested in.

1.2.2 Our Objectives

Our objectives for this puzzle were to first understand the solution of the problem and the mathematical concepts behind it, to investigate how variations of the problem affect the solution and lastly to find a general method to apply when solving variations of the problem.

1.2.3 Our Research Questions

Our Research Questions are:

1. What is the solution to the original problem and how does it work?
2. How does the number of the pins affect the solution of it falling after one pin is pulled? Is there a Formula?

3. What is the general rule to solving variations of the problem?

1.3 Scope of Study

For the classic knot problem, we will be finding a Formula that allows for solutions to this problem with any number of pins. We will also be looking into variants of this problem to search for a general rule or Formula to solve them.

1.4 Terminology

‘Cancelling out’ refers to when two sequences erase each other, in the form of a lowercase letter x being next to an uppercase letter X. As such, the sequence ‘aA’ cancels out.

The ‘inverse’ of a sequence is a sequence which when placed to the original sequence cancels both out, eg. the inverse of ‘a’ is ‘A’, the inverse of ‘aBCd’ is ‘DcbA’. As can be seen, aBCdDcbA will eventually cancel each other out.
Note that these inverses hold all the properties of the original sequence, such as if the original sequence would cancel out when pin A were to be removed, so would the inverse.

2 Literature Review

In 2004, a research paper written by Demaine E. D. proves that there is always a solution for problems like this. He shows through mathematical reasoning that problems in this way of removing a single pin amongst x pins always has a solution.
In 2014, another paper also by Demaine E. D. et. al. discusses the problem in-depth, demonstrating how such knots can be represented by a notation which we adapted to suit our needs.

A book written by Kranakis, E., Krizanc, D., Luccio (Eds.), F. in 2012 talks about the classic knot problem and possible variations to the puzzle. The variations we came up with were all based off the variations presented there.

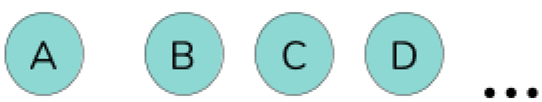
3 Methodology and Study

3.1 Methodology

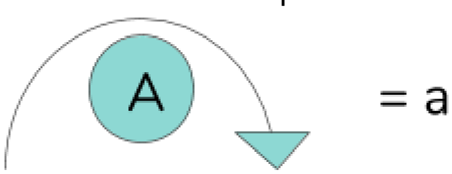
We created a system of notation for this problem such that any sequence of operations on the pins can be recorded into a single line of letter.

3.1.1 Our notation

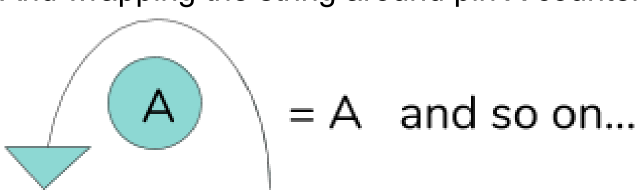
First, we label the pins pin A, pin B, pin C, pin D…



Then we label the operation of wrapping the string around pin X clockwise as “x”...

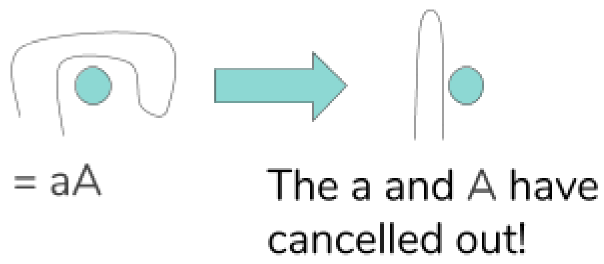


And wrapping the string around pin X counter-clockwise as “X”



With this notation, we are able to solve the problem.

We also noted that if there are 2 operations on a single pin where one of which is clockwise and the other of which is counter-clockwise, and are next to each other in a sequence, they will cancel out, as mentioned in our Terminology.



3.2 Results

By experimenting around with our notation, we have found answers to our research questions.

3.2.1 Results for Research Question 1

The sequence for the problem where there are 2 pins is “abAB”. This was derived through first inserting a loop around pin B in between the loops around pin A, forming ‘abA’, then adding another loop around pin B at the end of the sequence so as to cancel both loops out whenever any pin would be removed.

3.2.2 Results for Research Question 2

For n pins, where n > 1, the sequence that makes the painting fall after pulling any one of the pins is (sequence for 1 less pin) (new pin) (inverse of sequence for 1 less pin) (new pin [Capital]) The sequence where n = 1 is just “a”.
We derived this through the same concept that we were able to solve the original problem; the sequence and the inverse would cancel out if any of the pins in them were removed, or if they were put next to each other (i.e. the ‘new pin’ was removed).

3.2.3 Results for Research Question 3

For research question 3, we have decided to study 2 different variations.

3.2.3.1 Variation 1

Our first variation is hanging a painting on 3 pins and requiring 2 pins to be removed for the painting to drop. We realised when 1 pin is removed, the problem must reduce down to the original problem. Thus the answer is abcABC.

We demonstrated that for this problem with n pins and (n-1) pins to be removed, the solutions all equaled to $p_1p_2...p_{n-1}\, p_n P_1P_2...P_{n-1}\, P_n$ where p_n refers to the clockwise wrap around pin n and P_n refers to the counter-clockwise wrap around pin n.

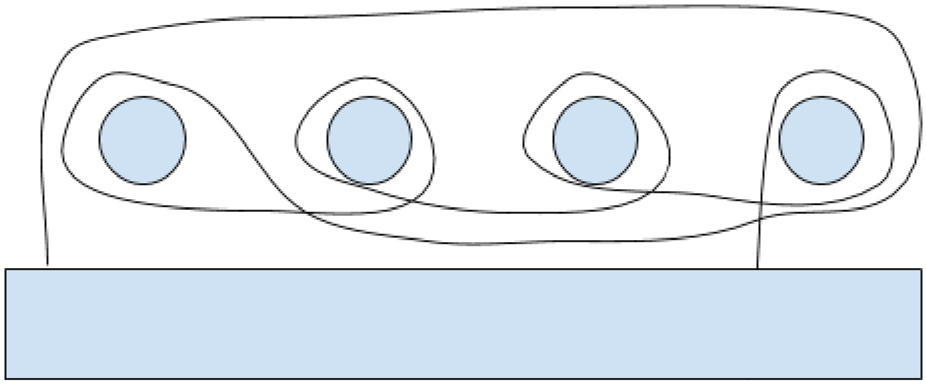
3.2.3.2 Variation 2

Our second variation is hanging a painting on 3 pins and requiring either both pins on the left or the rightmost pin to be removed before the painting falls. Let the left pins be A and B and the rightmost pin C. If we were to remove pin C, pin A and B must cancel out. This means that the sequence with A and B (i.e. ab) must also have an inverse separated by C. Thus the answer is abcBAC. If we were to represent the sequence in terms of n, it would be $p_1p_2...p_{n-1}p_nP_{n-1}P_{n-2}...P_2P_1P_n$.

3.2.3.3 The General Rule

A general rule we have found when solving for variations is that its solution can be derived from the original problem. For more complex problems, the answer can be derived from simpler variations of the base problem.

For variations of variation 1, let the number of pins to be removed equal to k. When k = n - 1, the starting of the sequence always starts with ‘a’ then continuing with the pins in order (abcd…) After this is the same thing but Capital. However, for the first section of lowercase, the string is just dragged above other pins. Essentially, this is a further extension of variation 1 and thus the answer should also follow. One example is one with 4 pins and 3 have to be pulled out before the painting drops.



4 Conclusion

There is a Formula to solve the classic knot problem, no matter how many pins there are. We have used a system of notation. Other variations of this problem can be either simplified into one of the classic knot problems, or are solved using a Formula.

5 References

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