The Polygon,

The Spiral,

The Mice

Leader:

LIU YIXUAN (3S120)

Members:

LOH CHI WEN (3S121)

HE DONGHANG (3S108)

LIU JUNCHENG (3O113)

CONTENTS

1. Introduction

1.1 Description

This project aims to research on The Mice Problem, otherwise known as The Beetles Problem. The original problem, which dates back to 1877, involved three beetles and an equilateral triangle. The description of the generalised problem is as follows: n mice start at the vertices of a regular n-gon and move towards the adjacent mouse in an anti-clockwise direction and meet at the centre of the figure. The movement of the mice to the centre of the circle is known as a pursuit curve. Much of the project would be aimed at investigating the properties and equation of the pursuit curve.

1.2 Objectives and Research Questions

1.2.1 Objectives

The objectives of this project are as follows:

- To obtain the general formula of the length of the pursuit curve in regular polygons.
- To find the general equation which describes the pursuit curve.
- To explore the properties of the pursuit curve
- Investigate the effects of changing variables and conditions

1.2.2 Research Questions

- What is the formula of the length of the pursuit curve?
- What is the equation that describes the pursuit curve?

- What is the equation of the pursuit curve when the $(m)th$ mouse moves towards the $(m+k)th$ mouse, where m and k are coprime?
- How will the path behave when the mice are replaced with frogs, with each frog jumping towards another frog at a discrete interval instead of moving continuously?

2. Literature Review

2.1 The Mice Problem

The Mice Problem, also known as The Beetle Problem, is when *n* mice start at the corners of a regular *n*-gon and move towards the adjacent mouse in a clockwise direction and meet at the centre of the figure. The problem is to solve the distance each mouse moves. The original problem, which had three beetles in an equilateral triangle moving to the centre, was first raised formally in 1877 and in 1880, Henri Brocard first proved that the trace of the beetle's movement is a logarithmic spiral.

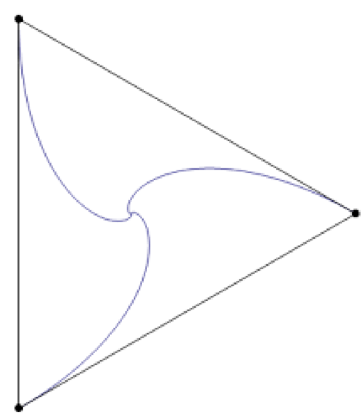

Figure 2.1 Pursuit Curve in an Equilateral Triangle

2.2 Simplifying the Problem by Introducing Polar Coordinate System

In Michael J. Seery's mathematics journal 'Pursuit and Regular N-gons', (Seery, 1998) the writer used polar coordinates to simplify the solution steps. This works because the polygons formed by the mice after each mouse moved a certain distance are symmetrical and each pursuit curve is formed by one mouse rotating in a spiral toward the centre of the polygon. The polar coordinate system which uses the distance from a reference point and an angle from a reference direction to determine the point can make the calculation more convenient.

2.3 Derive the Equation of the Pursuit Curve by Solving Differential Equation

The equation of the pursuit curve can be derived through differential equations as each mouse is moving towards the direction of a certain mouse. According to the physics knowledge of motion, we know that the speed direction of the object is always tangent to its track. This means the connecting line segment between each mouse and the mouse it is moving toward to is tangent to the curve. Hence, the slope of the tangent, which is also the derivative of the pursuit curve can be obtained. By solving this differential equation, the equation of the curve is derived.

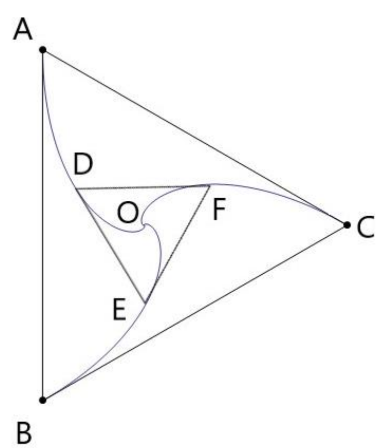

Figure 2.3.1

In the situation shown in this image, segment DE, EF and FD are tangent to curve ADO, BEO and CFO respectively.

3. Methodology

The following terminology and concepts are vital in the subsequent research conducted.

Term	Explanation
Regular polygon	In Euclidean geometry, a regular polygon is a polygon that is equiangular and equilateral.
Pursuit curve	A curve of pursuit is a curve constructed by analogy to having a point or points representing pursuers and pursuees, the curve of pursuit is the path taken by the pursuer.
Trigonometric function	In mathematics, trigonometric functions are functions of an angle. They relate the angles of a triangle to the lengths of its sides.
Derivative	The derivative is a ratio of change in the value of the function to change in the independent variable which measures the steepness of the graph of a function at some particular point on the graph.
Polar coordinate	In mathematics, the polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a

	distance from a reference point and an angle from a reference direction. The reference point (analogous to the origin of a Cartesian coordinate system) is called the pole, and the ray from the pole in the reference direction is the polar axis. The distance from the pole is the radius, and the angle is called the angular coordinate, polar angle, or azimuth.

3.1.1 Resolution of Vectors

Vectors are physics quantities which have both direction and magnitude. In physics, it is said that vectors can be decomposed or resolved into two or several components whose sum is the original vector. The resolution of two vectors, which is what we will be used for solving Research Question 1 is the resolution of two vectors. It can be achieved by applying the parallelogram method. The parallelogram method states that any vector can be resolved into two parts, each of which being one side of the parallelogram and the original vector being the diagonal of the parallelogram.

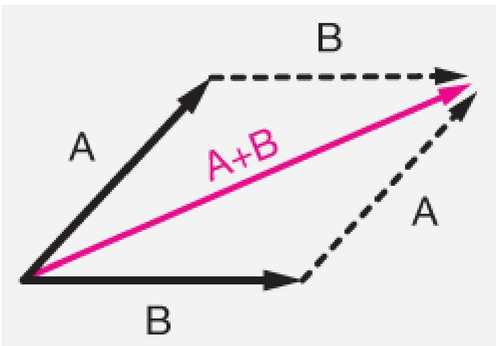

Figure 3.1.1

3.1.2 Calculus Formulas

In this project, the followed differential properties have been utilised.

Property Name	Formula
Product Rule	$h'(x) = (fg)'(x)$ $= f'(x)g(x) + f(x)g'(x)$
Chain Rule	The derivative of the function $h(x) = f(g(x))$ with respect to x is as follows $h'(x) = f'\big(g(x)\big) \cdot g'(x)$

3.1.3 Trigonometric Functions

In this project, the following trigonometric functions have been utilised.

Formula
$\frac{d}{dx}\sin x = \cos x$
$\frac{d}{dx}\cos x = -\sin x$

4 Studies and Solutions

4.1 Research Question 1

A physics method was used to approach and solve this research question. The method applied was the resolution of vectors.

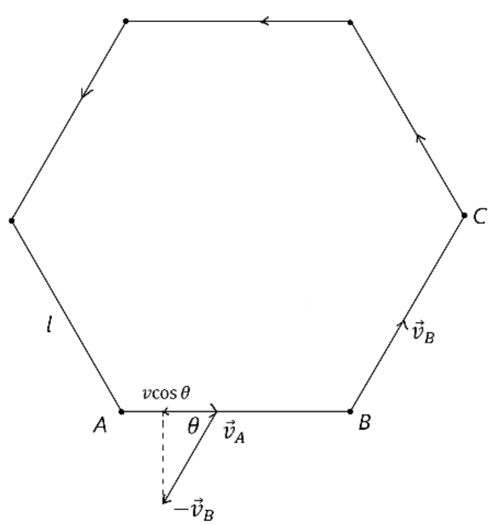

Figure 4.1.1

Let the length of the sides of the regular polygon be l,

the magnitude of the velocity of the points be v,

the exterior angle of the polygon be θ.

Since rotational symmetry exist between the polygons formed by the pursuit curve, only the properties of one of the pursuit curves require investigation. In the above example, the left bottom point, A, was utilised as the starting point and the mice at A will move towards the mouse at B. To simplify the calculation, point B was used as the reference point. Hence, the relative velocity of A to B is $\vec{v}_{AB} = \vec{v}_A + (-\vec{v}_B)$. By resolving the velocity of $-\vec{v}_B$ into the horizontal portion, which is in the direction of B, and the vertical portion, the magnitude of the velocity of A in the direction of B can be obtained as $v_{AB} = v - v\cos\theta$. The time taken for the points to meet at the centre of the polygon is exactly the time taken for segment AB to decrease to 0. Hence, the time taken for the whole process is

$$t = \frac{AB}{v_{AB}} = \frac{l}{v - v\cos 0} = \frac{l}{v(1 - \cos\theta)}$$

Since distance is velocity multiplied by time, hence, the length of the curve is

$$s = vt = \frac{l}{1 - \cos\theta}$$

Since θ is the exterior angle of the polygon, $\theta = \frac{360^0}{n}$. Thus, the length formula of the pursuit curve in a regular polygon with n sides is

$$S = \frac{l}{1 - \cos\frac{360^0}{n}}$$

Below is a table of some n values and the corresponding S values.

n values (n being the number of sides)	S value (rounded to 3s.f.)
3	0.667
4	1.00
5	1.48
6	2.00

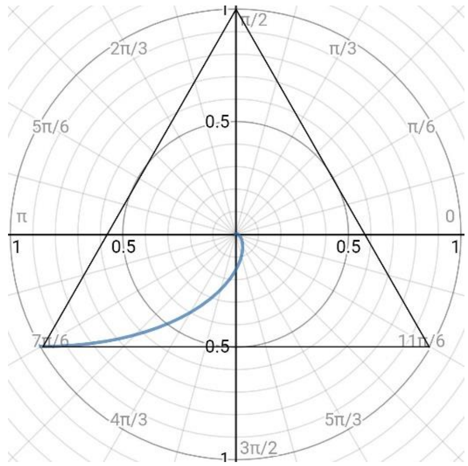

Figure 4.1.2

When $n = 3$, the above graph is formed.

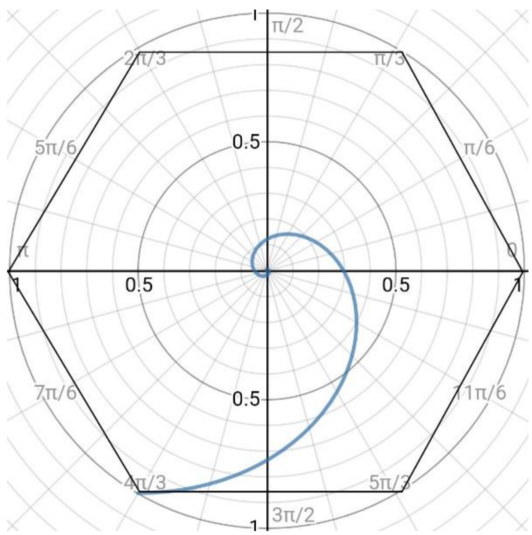

Figure 4.1.3

When $n = 6$, the above graph is formed.

Some observations include as the n value increase, the S value will also increase.

Note that the length of each side of the polygon is always assumed to be one.

4.2 Research Question 2

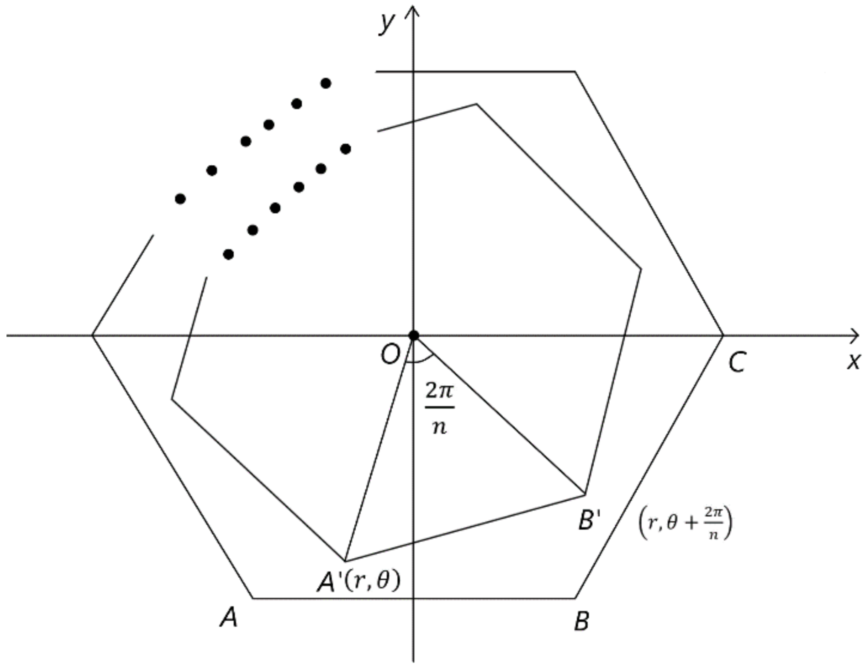

Figure 4.2.1

In a generalised case, the polar coordinates of A' would be (r, θ) and the coordinates of B' would be $\left(r, \theta + \frac{2\pi}{n}\right)$. The corresponding coordinates of A' and B' in Cartesian plane would be $A'(r\cos\theta, r\sin\theta)$ and $B'\left(r\cos\left(\theta + \frac{2\pi}{n}\right), r\sin\left(\theta + \frac{2\pi}{n}\right)\right)$. The gradient of line $A'B'$ can be subsequently obtained as $m_{A'B'} = \frac{r\sin\left(\theta+\frac{2\pi}{n}\right) - r\sin\theta}{r\cos\left(\theta+\frac{2\pi}{n}\right) - r\cos\theta}$.

Through simplification and manipulation using trigonometric properties, a simpler form of the expression can be obtained.

$$m_{A'B'} = \frac{\left(\cos\frac{2\pi}{n} - 1\right)\sin\theta + \sin\frac{2\pi}{n}\cos\theta}{-\sin\frac{2\pi}{n}\sin\theta + \left(\cos\frac{2\pi}{n} - 1\right)\cos\theta}$$

For simpler calculation, let $k_1 = \cos\frac{2\pi}{n} - 1$ and $k_2 = \sin\frac{2\pi}{n}$. A new version of the expression is produced.

$$m_{A'B'} = \frac{k_1 \sin\theta + k_2 \cos\theta}{-k_2 \sin\theta + k_1 \cos\theta}$$

The slope of the line is also equal to the derivative of the pursuit curve

$$\frac{dy}{dx} = \frac{\mathrm{r}' \cdot \sin\theta + \mathrm{r} \cdot (\sin\theta)'}{r' \cdot cos\theta + r \cdot (cos\theta)'}$$

$$\frac{dy}{dx} = \frac{\sin\theta\, dr + r\cos\theta\, d\theta}{\cos\theta\, dr - r\sin\theta\, d\theta}$$

Hence, the differential equation is constructed:

$$\frac{dy}{dx} = m_{A'B'}$$

$$\frac{\sin\theta\, dr + r\cos\theta\, d\theta}{\cos\theta\, dr - r\sin\theta\, d\theta} = \frac{k_1 \sin\theta + k_2 \cos\theta}{-k_2 \sin\theta + k_1 \cos\theta}$$

$$k_2 \frac{1}{r} dr = k_1\, d\theta$$

$$\int \frac{1}{r} dr = \int \frac{k_1}{k_2} d\theta$$

$$\ln r = \frac{k_2}{k_1}\theta + C$$

Because the curve passes $A\left(1, \frac{3}{2}\pi - \frac{\pi}{n}\right)$, when $r = 1$, $\theta = \frac{3}{2}\pi - \frac{\pi}{n}$. Thus, $C = \frac{k_2}{k_1}\left(\frac{\pi}{n} - \frac{3}{2}\pi\right)$. The final equation for the pursuit curve in a regular polygon with n sides is therefore

$$r = e^{\frac{\cos\frac{2\pi}{n} - 1}{\sin\frac{2\pi}{n}}\theta + \frac{\cos\frac{2\pi}{n} - 1}{\sin\frac{2\pi}{n}}\left(\frac{\pi}{n} - \frac{3}{2}\pi\right)}$$

Below are some examples of the curves formed for varying n values.

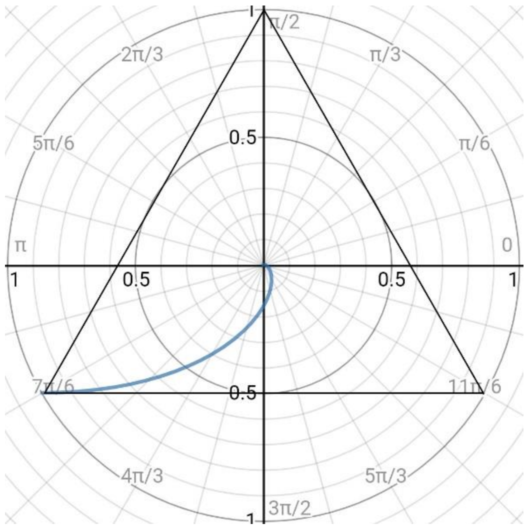

Figure 4.2.2 When $n = 3$

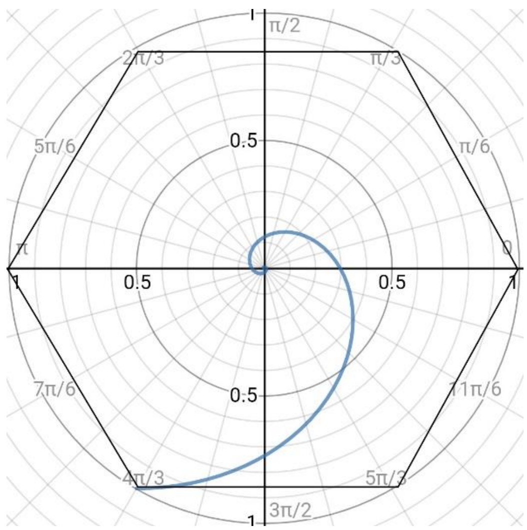

Figure 4.2.3 When $n = 6$

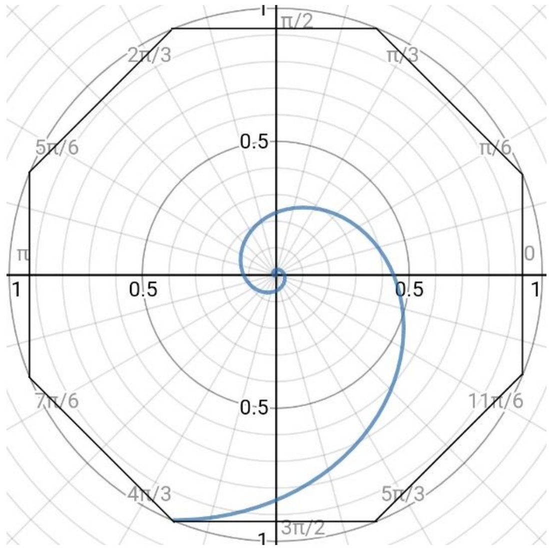

Figure 4.2.4 When $n = 8$

Observation made: As n value increases, the length of the curve and the curvature will increase.

4.3 Solution to Research Question 3

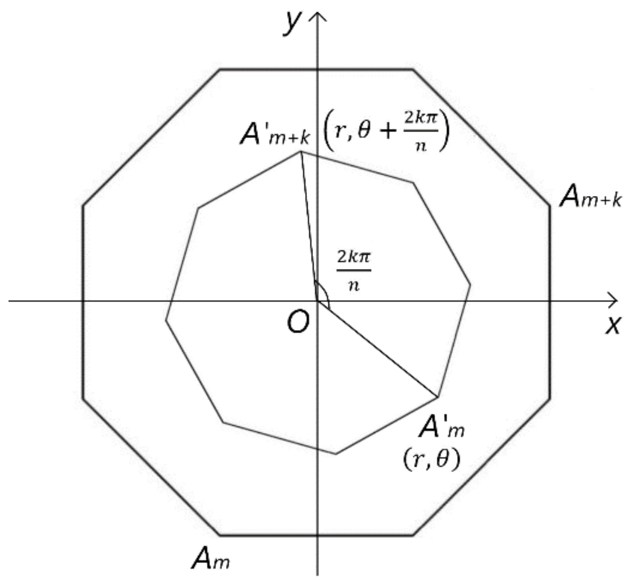

Figure 4.3.1

The difference of angle between the $(m)th$ mice and the $(m+k)th$ mice is $\frac{2k\pi}{n}$, with $k = y + 1$ and $y =$ number of mice between the $(m)th$ mouse and $(m+k)th$ mouse. In the previous Research Question, $k = 1$ and angle between each two adjacent mice is $\frac{2\pi}{n}$.

Due to this, the solution for this problem would be similar to the previous problem.

Let the coordinates of A' be (r, θ) and the coordinates of B' be $\left(r, \theta + \frac{2k\pi}{n}\right)$.

The coordinates of A' and B' in Cartesian coordinates would be $A'(r\cos\theta, r\sin\theta)$ and $B'\left(r\cos\left(\theta + \frac{2\pi}{n}\right), r\sin\left(\theta + \frac{2\pi}{n}\right)\right)$.

The slope of $A'B'$ is

$$m_{A'B'} = \frac{\left(\cos\frac{2k\pi}{n} - 1\right)\sin\theta + \sin\frac{2k\pi}{n}\cos\theta}{-\sin\frac{2k\pi}{n}\sin\theta + \left(\cos\frac{2k\pi}{n} - 1\right)\cos\theta}$$

The derivative of the pursuit curve is

$$\frac{dy}{dx} = \frac{\sin\theta\, dr + r\cos\theta\, d\theta}{\cos\theta\, dr - r\sin\theta\, d\theta}$$

$$\frac{dy}{dx} = m_{A'B'}$$

$$\frac{\sin\theta\, dr + r\cos\theta\, d\theta}{\cos\theta\, dr - r\sin\theta\, d\theta} = \frac{\left(\cos\frac{2k\pi}{n} - 1\right)\sin\theta + \sin\frac{2k\pi}{n}\cos\theta}{-\sin\frac{2k\pi}{n}\sin\theta + \left(\cos\frac{2k\pi}{n} - 1\right)\cos\theta}$$

$$\frac{\sin\frac{2k\pi}{n}}{r}dr = \cos\frac{2k\pi}{n} - 1\, d$$

$$\int \frac{\sin\frac{2k\pi}{n}}{r}dr = \int \left(\cos\frac{2k\pi}{n} - 1\right) d\theta$$

$$r = e^{\frac{\cos\frac{2k\pi}{n}-1}{\sin\frac{2k\pi}{n}}\theta + \frac{\cos\frac{2k\pi}{n}-1}{\sin\frac{2k\pi}{n}}\left(\frac{\pi}{n} - \frac{3}{2}\pi\right)}$$

The equation shown above is the one that describes the pursuit curve.

Below is a case when $n = 31$ and k ranging from 1 to 8.

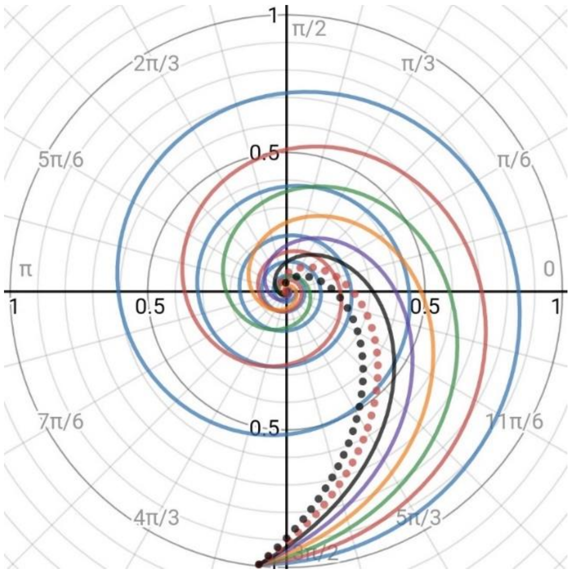

Figure 4.3-2

Line Type	Blue Solid Line	Pink Solid Line	Green Solid Line	Orange Solid Line	Purple Solid Line	Black Solid Line	Pink Dotted Line	Black Dotted Line
K value	1	2	3	4	5	6	7	8

Observations: When n is set and k increases, the length and curvature of the curve decrease.

4.4 Solution to Research Question 4

In this Research Question, the situation being investigated is where each frog jumps towards its immediate neighbour in an anti-clockwise direction at a constant jump length. A computer program was designed to show the results of this question and facilitate the further investigations which led to a legitimate mathematical proof of the computer-generated result.

4.4.1 Utilising Computer Program to Find the Result

Since this question is too complicated to visualise and calculate manually, a computer program was designed to facilitate the research of the final results.

If the frogs were to jump in some sort of loop eventually, the distance between every two frogs at last will be exactly the same as the length of each jump. Hence, by calculating the distance between every two adjacent frogs and compare it to the length of each jump, whether the loop is formed can be determined.

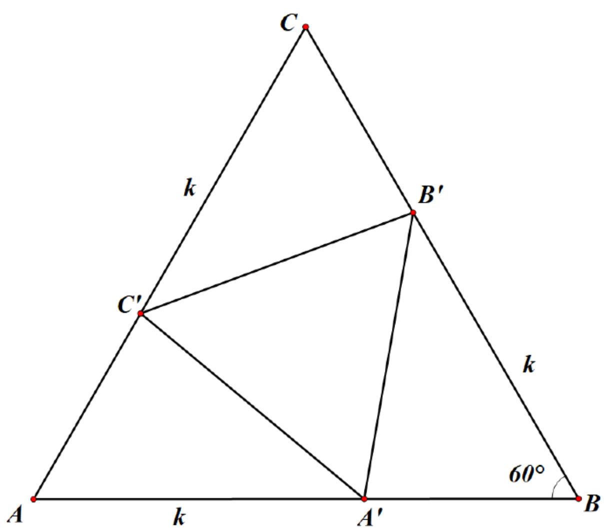

Figure 4.4-1

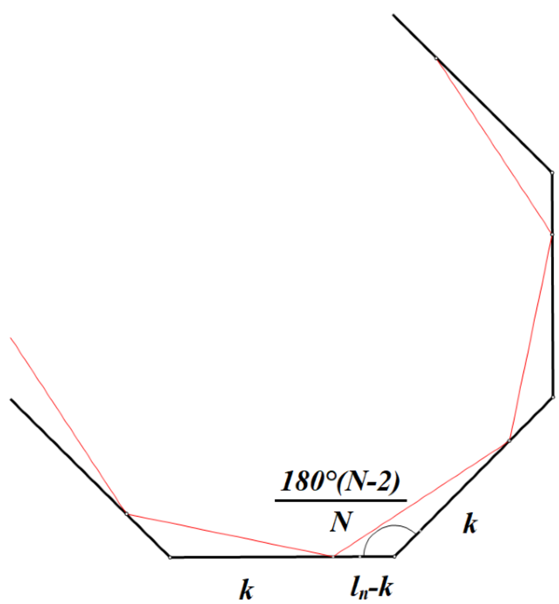

Figure 4.4-2

Let the length of each jump by frogs be k and the side length after the $n\,th$ jump be l_n. Note that the number of sides of the regular polygon is denoted as N.

By applying the Cosine Law in the triangle formed by the sides of the new triangle and the previous triangle, the length of each new triangle formed by the frogs can be determined as follow:

$$l_{n+1} = \sqrt{k^2 + (l_n - k)^2 - 2k(l_n - k)\cos\left[\frac{180(N-2)}{N}\right]}\ (k < l_n)$$

By designing a computer program to calculate this formula, the lengths of the sides of the triangles formed by the frogs in a regular N-gon after every step can be obtained. By analysing the data obtained from the computer program, we were able to discover that no matter what the starting jump length is, a loop will always form eventually in any regular polygon.

4.4.2 The Mathematical Proof of the Result

Since the result obtained from the computer program is that a loop will always form in any regular polygon, the mathematical proof should aim to prove that the difference between the side length of the regular polygon and the jump length will always decrease such that after infinite times of jumps, the frogs will be at a determined position and jump in a loop. Depending on the internal angle of the regular polygon, the proof of the result is divided into two major parts, one for equilateral triangle whose internal angles are smaller than 90^0 which will complicate the motion, and regular polygons with more than or equal to 4 sides whose internal angles are greater than or equal to 90^0 which are simpler cases.

4.4.2.1 The Case in Equilateral Triangle

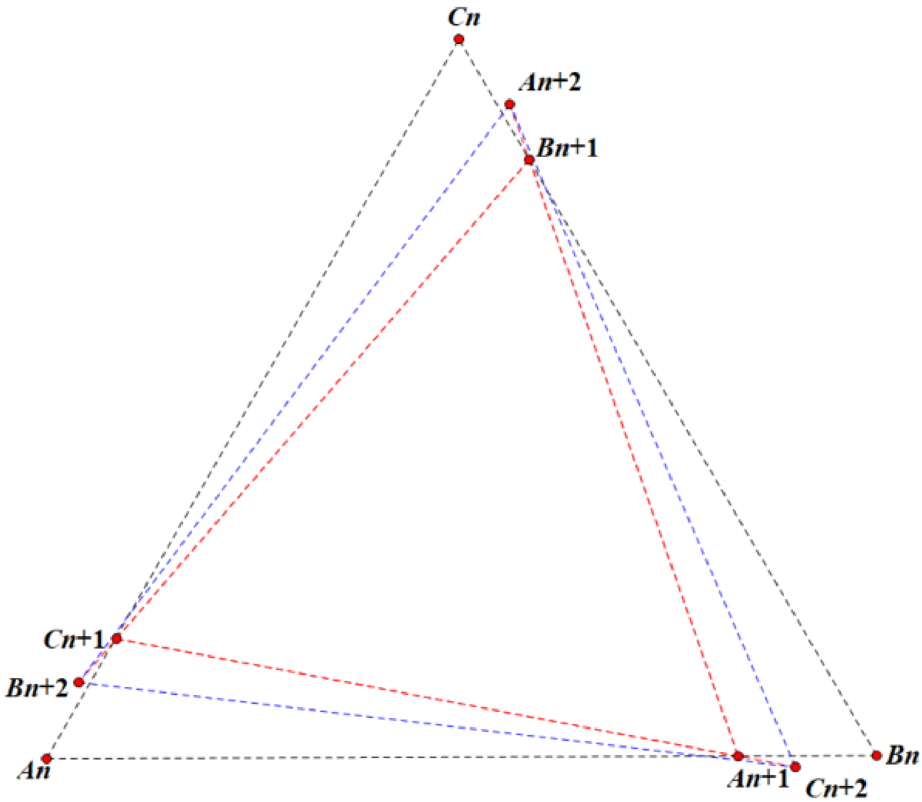

Figure 4.4-3

Three frogs start from a triangle whose length is set to be 1 and they jump towards their respective immediate neighbour in an anti-clockwise direction. As the frogs continue to jump, it is obvious that the size of the triangle formed by the three frogs will decrease gradually. At a particular point of time (let it be the $n\,th$ jump), the length of each jump k, and the distance between each two adjacent frogs l_n will have this following relation:

$$\frac{1}{2}l_n \leq k < l_n$$

Firstly, if $k = \frac{1}{2}l_n$, then the frogs will immediately jump into a loop at the next jump and the triangle formed by them will thus be determined.

If $\frac{1}{2}l_n < k < l_n$, we will select the frog at point A_n to study its motion and name it Frog A.

At the following jump, Frog A will be at the position A_{n+1} and same for the other two frogs who will be at position B_{n+1} and C_{n+1}.

Consider the triangle $A_{n+1}B_nB_{n+1}$ which contains the two sides which have the side length of the new triangle formed by the three frogs segment $A_{n+1}B_{n+1}$ and the jump length segment B_nB_{n+1}.

Since $\angle B_nA_{n+1}B_{n+1} > \angle A_{n+1}B_nB_{n+1}$, we can conclude that

$$B_nB_{n+1} > A_{n+1}B_{n+1}$$

which indicates that the new triangle has a smaller side length that that of each jump. Hence, in the next jump, the frogs will jump out of the current triangle to a position on the extension of the sides of the current triangle.

The new triangle $A_{n+2}B_{n+2}C_{n+2}$ is then obtained after the $(n+2)th$ jump. In this new triangle, consider the triangle $A_{n+2}B_{n+1}B_{n+2}$ which contains the two important length, the side length of the new triangle segment $A_{n+2}B_{n+2}$ and the jump length segment $B_{n+1}B_{n+2}$. The following properties of the angles in this particular triangle is obtained:

$$\angle A_{n+2}B_{n+1}B_{n+2} > \angle B_{n+1}A_{n+2}B_{n+2}$$

Hence, we can conclude that

$$A_{n+2}B_{n+2} > B_{n+1}B_{n+2}$$

which indicates that the side length of the new triangle is greater than the jump length. Therefore, in the next step, the frogs will not jump outside of the side of the triangle.

Since

$$\Delta A_{n+1}A_{n+2}C_{n+2} \cong \Delta B_{n+1}B_{n+2}A_{n+2} \cong \Delta C_{n+1}C_{n+2}B_{n+2}$$

we can obtain that

$$\angle A_{n+2}B_{n+2}B_{n+1} = \angle A_{n+1}A_{n+2}C_{n+2}$$

and because $\angle B_{n+2}A_{n+2}C_{n+1} = 60^0$, we can obtain that $\angle A_{n+2}B_{n+1}B_{n+2} = 120^0$.

Because

$A_{n+2}A_{n+2} = B_{n+1}B_{n+2}$ and $A_{n+2}B_{n+1} = A_{n+1}A_{n+2} - A_{n+1}B_{n+1}$

we can obtain

$$A_{n+2}B_{n+1} < B_{n+1}B_{n+2}$$

$$30^0 < \angle B_{n+1}A_{n+2}B_{n+2} < 60^0$$

By applying the Sine Law,

$$\frac{\sin \angle A_{n+2}B_{n+1}B_{n+2}}{A_{n+2}B_{n+2}} = \frac{\sin \angle B_{n+1}A_{n+2}B_{n+2}}{B_{n+1}B_{n+2}}$$

$$\frac{\sin 120^0}{l_{n+2}} = \frac{\sin \angle B_{n+1}A_{n+2}B_{n+2}}{k}$$

$$k = \frac{\sin \angle B_{n+1}A_{n+2}B_{n+2}}{\sin 120^0} l_{n+2}$$

Since

$$\sin 120^0 = \frac{\sqrt{3}}{2}$$

$$30^0 < \angle B_{n+1}A_{n+2}B_{n+2} < 60^0$$

$$\frac{1}{2} < \sin \angle B_{n+1}A_{n+2}B_{n+2} < \frac{\sqrt{3}}{2}$$

$$\frac{1}{2} l_{n+2} < \frac{\sqrt{3}}{3} l_{n+2} \angle k < l_{n+2}$$

Therefore, the jump length k satisfies

$$\frac{1}{2}l_{n+2} < k < l_{n+2}$$

Which indicates that the next jump (i.e. the $(n+3)th$ jump), will be exactly the same condition as that of the $(n+1)th$ jump. Hence, it is proven that starting from the $(n+1)th$ jump, the jump length will be smaller than and then greater than the side length of the triangle formed by the three frogs alternately.

Therefore, we can conclude that for every l_{n+i}, when i is odd, the side length is smaller than the jump length, when i is even, the side length is greater than the jump length.

3.2.4.2 Prove the reduction of the difference between the jump length k and the side length l_n

Let Δx_n represents the length difference between the side length of the triangle formed by the three frogs and the length of each jump.

$$\Delta x_n = |l_n - k|$$

When the three frogs are at positions $A_n, B_n \; and \; C_n$ the side length is greater than the jump length.

i.e. $$l_n > k$$

Therefore,

$$\Delta x_n > \; ln - k$$

After the $(n+1)th$ jump and the new triangle is formed, consider the triangle $A_{n+1}B_nB_{n+1}$ and the following condition can be obtained from previous proven:

$$\angle A_{n+1}B_nB_{n+1} = 60^0$$

$$B_nB_{n+1} = k$$

$$A_{n+1}B_{n+1} = l_{n+1}$$

By applying the cosine law

$$l_{n+I}^2 = (l_n - k)^2 + k^2 - 2(l_n - k)k \cos 60^0$$

Expand and simplify

$$l_{n+1} = \sqrt{3k^2 - 3l_n k + l_n^2}$$

In the $(n+1)th$ jump, the side length formed by the three frogs will be smaller than the jump length.

Hence

$$\Delta x_n = |l_{n+1} - k|$$

$$= k - \sqrt{3k^2 - 3l_n k + l_n^2}$$

After the $(n+2)th$ jump and the new triangle is formed, consider the triangle $B_{n+1}A_{n+2}B_{n+2}$, we can obtain that

$$\angle B_{n+1}A_{n+2}B_{n+2} = 120°$$

$$B_{n+1}B_{n+2} = k$$

$$A_{n+2}B_{n+2} = l_{n+2}$$

By applying the cosine law, $l_{n+2}^2 = (k - l_{n+1})^2 + k^2 - 2(k - l_{n+1})k\cos 120°$

Expand and simplify

$$l_{n+2} = \sqrt{3k^2 - 3l_{n+1}k + l_{n+1}^2}$$

In the $(n+2)th$ jump, the side length formed by the three frogs will be larger them the jump length.

Hence,

$$\Delta X_{n+2} = |l_{n+2} - k|$$

$$= \sqrt{3k^2 - 3l_{n+1}k + l_{n+1}^2} - k$$

Compare the difference between Δx_n and ΔX_{n+2}

$$\Delta X_{n+2} - \Delta x_n = \left(\sqrt{3k^2 - 3l_{n+1}k + l_{n+1}^2} - k\right) - (l_{n+1} - k)$$

$$= \sqrt{3k^2 - 3l_{n+1}k + l_{n+1}^2} - l_{n+1}$$

Since it is difficult to tell whether this expression is positive or negative we can compare the two terms $\sqrt{3k^2 - 3l_{n+1}k + l_{n+1}^2}$ and l_{n+1}

$$\left(\sqrt{3k^2 - 3l_{n+1}k + l_{n+1}^2} - k\right)^2 - l_n^2 = 3k^2 - 3l_{n+1}k + l_{n+1}^2 - l_n^2$$

Substitute $l_{n+1} = \sqrt{3k^2 - 3l_nk + l_n^2}$

$$\left(\sqrt{3k^2 - 3l_{n+1}k + l_{n+1}^2} - k\right)^2 - l_n^2 = 3k^2 - 3l_{n+1}k + l_{n+1}^2 - l_n^2$$

$$= 6k^2 - 3l_{n+1}k - l_n^2$$

$$= 3k(2k - l_{n+1} - l_n$$

Since, $k < l_n$ and $k < l_{n+1}$

We can obtain that the value of the expression is always negative. Therefore, the reduction of the difference between side length and jump length is proven.

4.4.3.1 The Case in Regular N-gon ($N \geq 4$)

In this case, the jump length is smaller than the side length of the regular polygon.

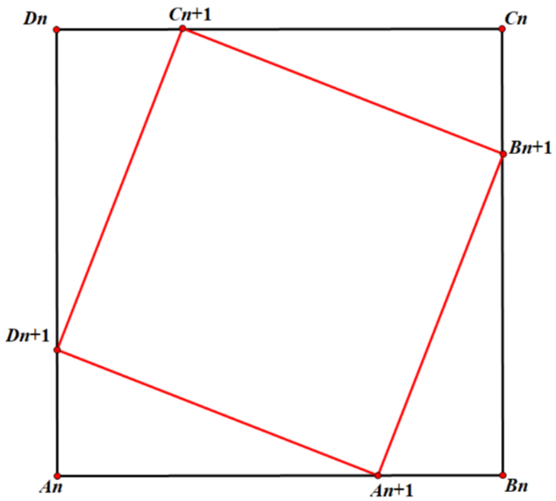

Figure 4.4-4

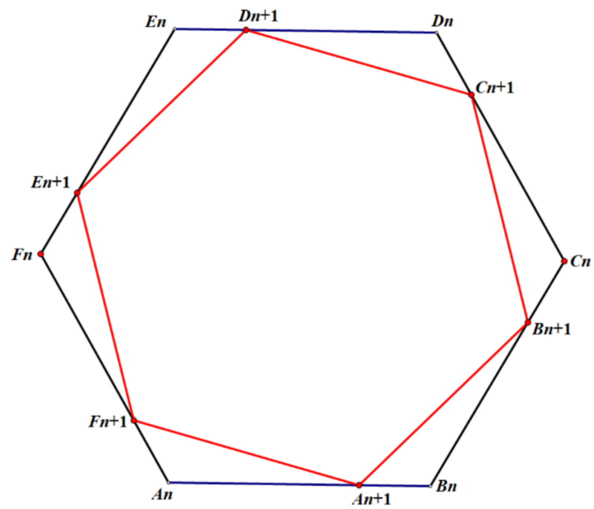

Figure 4.4-5

In the regular N-gon with $N \geq 4$, the following information can be obtained (note that square is used here to illustrate the proof but the proof for any other regular polygons should be the same):

$$l_n = A_n B_n$$

$$l_{n+1} = A_{n+1} B_{n+1}$$

$$k = B_n B_{n+1}$$

Consider the triangle $A_{n+1} B_{n+1} B_n$

The angles of the regular N-gon can be calculated

$$\angle A_{n+1} B_n B_{n+1} = \frac{180(N-2)}{N} = 180° - \frac{360}{N}$$

When $N \geq 4$

$180° - \frac{360}{N} \leq 90°$

Hence, $\angle A_{n+1} B_n B_{n+1} \geq 90°$ in any regular N-gon

This indicates that $\angle A_{n+1}B_nB_{n+1}$ is always the greatest angle in $\Delta A_{n+1}B_nB_{n+1}$

Therefore, the following relation is obtained:

$$l_{n+1} \geq k$$

Thus, we can deduce that the side length is always greater than the jump length in any regular N-gon

$\because$ $\Delta A_{n+1}B_nB_{n+1} \cong \Delta A_{n+1}A_nD_{n+1}$

$\therefore$ $l_n = A_{n+1}B_n + B_nB_{n+1}$

$\because$ $A_{n+1}B_{n+1} < A_{n+1}B_n + B_nB_{n+1}$

$\therefore$ $l_{n+1} < l_n$

Therefore, the reduction of the side lengths of the regular polygons formed by the frogs is proven. The lengths will continuously reduce after each jump until they are infinitely close to the jump length and form loops.

When the jump length is greater than the side length in the polygon, where $N \geq 4$

By using the previous method, we obtain the following equations

$$l_1 = \sqrt{k^2 + (l_0 - k)^2 - 2k(l_0 - k)\cos\left(\frac{180(N-2)}{N}\right)}$$

$$= \sqrt{2k^2 - 2kl_0 + l_0^2 + (2k^2 - 2kl_0)\cos(\frac{180(N-2)}{N})}$$

It is obvious that $l_1 \geq l_0$,

Similarly, we can obtain that when $N \geq 6$, $k > l_1$

When $4 \leq N \leq 6$ $k \leq l_1$,

$$l_2 = \sqrt{k^2 + (l_1 - k)^2 - 2k(l_1 - k)\cos(\frac{180(N-2)}{N})}$$

$$= \sqrt{2k^2 - 2kl_1 + l_1^2 + (2k^2 - 2kl_1)\cos(\frac{180(N-2)}{N})}$$

Similarly, we can prove that $l_2 \leq l_1$,

$$\Delta x_0 = |l_0 - k|$$

$$\Delta x_1 = |l_1 - k|$$

$$\Delta x_2 = |l_2 - k|$$

$\because l_0 < k, l_2 < k$

$\therefore$ $$\Delta x_0 - \Delta x_2 = l_2 - l_0$$

$$\Delta x_0^2 - \Delta x_2^2 = l_2^2 - l_0^2$$

$$l_0^2 - l_2^2 = (k - l_1)\left\{k - l_1 + 2kcos\left[\frac{180(\mathrm{n}-2)}{\mathrm{n}}\right]\right\} + (\mathrm{k} - l_1)$$

$\because$ $(k - l_1) \leq 0$, $\left\{k - l_1 + 2kcos\left[\frac{180(\mathrm{n}-2)}{\mathrm{n}}\right]\right\} \leq 0, +(\mathrm{k} - l_1) \geq 0$

$\therefore$ $l_0^2 \geq l_2^2$

Therefore, the difference between the jump length and the side length of the regular polygon will decrease and loops will eventually form in any regular polygon.

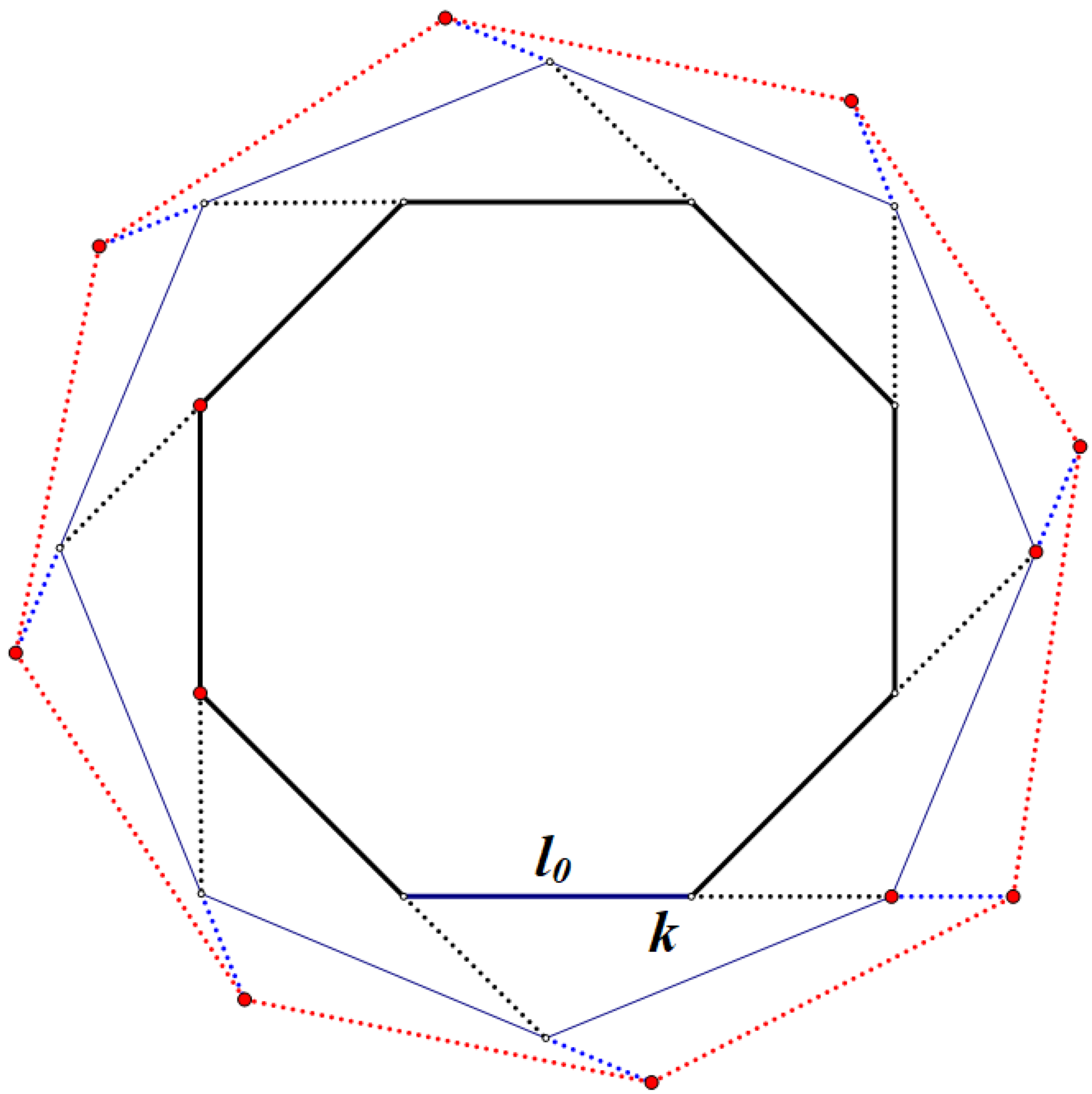

Figure 4.4.6

5. Conclusion

In conclusion, the project has ultimately achieved its aims of investigating the properties of the pursuit curve in a regular polygon and the effects of various differing conditions. Various equations that describes various curves have been created for the respective differing conditions and a formula to find the length of the curve has also been found. Last but not least, the proof of the complicated discrete mathematics variation of the pursuit curve where each frog is moving in a discrete manner with constant jump length and form loops eventually is found.

Because of the limited time and mathematics knowledge, this research has its own limitations. For Research Question 4 where the continuous movement of the mice is replaced by the discrete jump of frogs, only the case of constant jump length is discussed although it is the most complicated case among all the other possible calculatable cases. Other variation of Research Question 4 includes jumping with changing but predictable jump length such as jumping at a changing jump length which is related to the number of jumps. For the case discussed of the constant jump length, there is also limitation. For example, the function which can describe the position of the frogs at each jump is not found. Considering that the difference between the jumping length and the side length of the regular polygon is ever decreasing, it can be deduced that the frogs will end up at some particular points when the loops are formed. However, the relationship between the jump length and the final position of the frogs are not found due to lack of mathematics knowledge and researching time.

6. References

1. Nahi, K., Kuch nahikuch nahi 3, Will JagyWill Jagy 108k55 gold badges105105 silver badges208208 bronze badges, Jim BelkJim Belk 38.7k22 gold badges8989 silver badges156156 bronze badges, DeepakDeepak 2111 bronze badge, DeechitDeechit 1, & Nprasnpras 111 bronze badge. (1961, December 01). The vertices of an equilateral triangle are shrinking towards each other. Retrieved March 11, 2019, from https://math.stackexchange.com/questions/44896/the-vertices-of-an-equilateral-triangle-are-shrinking-towards-each-other

2. (n.d.). Retrieved July 30, 2019, from https://www.mathpages.com/home/kmath492/kmath492.htm

3. Mice problem. (2019, May 24). Retrieved July 30, 2019, from https://en.wikipedia.org/wiki/Mice_problem

4. Mice Problem. (n.d.). Retrieved July 30, 2019, from http://mathworld.wolfram.com/MiceProblem.html

APPENDIX

I. Code Utilised in Research Question 4

```
#include <stdio.h>
#include <math.h>
#include <limits.h>
#include <stdlib.h>
#include <time.h>

int main() {
	double arr[10000];
	int i = 0;
	long count = 0;
	double k = 0;
	time_t t;
	double angle;

	for (i = 0; i < 10000; i++)
		arr[i] = 0;
	arr[0] = 1;
	srand((unsigned) time(&t));
	k = rand();
	srand((unsigned) time(&t));
	angle = rand() % 120 + 60;
	printf("angle is %lf\n", angle);
	while (k >= 3){
		k /= rand();
	}
       printf("the value of k is %lf\n", k);
	i = 0;
	while ( i < 1000){
//		printf("%lf", pow((arr[i] - k), 2.0));
//		printf("%")
		arr[i + 1] = sqrt(pow((arr[i] - k), 2.0) + k * k - 2.0 * (arr[i] - k) * k * (cos (angle)));
```

```
        printf("the number becomes %lf\n", arr[i + 1]);
        i++;
        count++;

    }

    printf("count is %ld\n", count);

    return 0;
}
```

II. Data Sets Obtained by the Code

The following data is obtained from the computer program. a few sets of data are obtained for some regular polygons to show the results of the Research Question 4. The originally distance between the frogs is set to be 1.

angle is 60					
the value of k is 0.898332					
0.995644	0.909191	0.899443	0.898445	0.898344	0.898334
0.991457	0.908681	0.89939	0.898439	0.898343	0.898333
0.987434	0.908194	0.89934	0.898434	0.898343	0.898333
0.983569	0.907729	0.899292	0.898429	0.898342	0.898333
0.979857	0.907287	0.899246	0.898425	0.898342	0.898333
0.976294	0.906865	0.899203	0.89842	0.898341	0.898333
0.972874	0.906462	0.899161	0.898416	0.898341	0.898333
0.969593	0.906079	0.899122	0.898412	0.89834	0.898333
0.966446	0.905713	0.899084	0.898408	0.89834	0.898333
0.963429	0.905365	0.899049	0.898405	0.89834	0.898333
0.960536	0.905033	0.899015	0.898401	0.898339	0.898333
0.957764	0.904716	0.898982	0.898398	0.898339	0.898333
0.955107	0.904415	0.898951	0.898395	0.898339	0.898333
0.952563	0.904127	0.898922	0.898392	0.898338	0.898333
0.950126	0.903853	0.898894	0.898389	0.898338	0.898333
0.947793	0.903592	0.898867	0.898386	0.898338	0.898333
0.945559	0.903343	0.898842	0.898384	0.898338	0.898333
0.943422	0.903106	0.898817	0.898381	0.898337	0.898333

0.941376	0.90288	0.898794	0.898379	0.898337	0.898333
0.93942	0.902665	0.898772	0.898377	0.898337	0.898333
0.937548	0.902459	0.898751	0.898375	0.898337	0.898333
0.935758	0.902264	0.898731	0.898373	0.898336	0.898333
0.934047	0.902078	0.898712	0.898371	0.898336	0.898333
0.932411	0.9019	0.898694	0.898369	0.898336	0.898333
0.930847	0.901731	0.898677	0.898367	0.898336	0.898333
0.929353	0.90157	0.898661	0.898366	0.898336	0.898333
0.927925	0.901416	0.898645	0.898364	0.898336	0.898333
0.92656	0.90127	0.89863	0.898362	0.898335	0.898333
0.925257	0.901131	0.898616	0.898361	0.898335	0.898333
0.924012	0.900998	0.898603	0.89836	0.898335	0.898333
0.922823	0.900871	0.89859	0.898358	0.898335	0.898333
0.921688	0.900751	0.898577	0.898357	0.898335	0.898333
0.920604	0.900636	0.898566	0.898356	0.898335	0.898333
0.91957	0.900527	0.898555	0.898355	0.898335	0.898333
0.918582	0.900423	0.898544	0.898354	0.898335	0.898333
0.917639	0.900323	0.898534	0.898353	0.898334	0.898333
0.916739	0.900229	0.898524	0.898352	0.898334	0.898333
0.91588	0.900139	0.898515	0.898351	0.898334	0.898333
0.915061	0.900053	0.898507	0.89835	0.898334	0.898333
0.914279	0.899971	0.898498	0.898349	0.898334	0.898333
0.913533	0.899893	0.89849	0.898348	0.898334	0.898333
0.912822	0.899819	0.898483	0.898348	0.898334	0.898333
0.912143	0.899749	0.898476	0.898347	0.898334	0.898333
0.911495	0.899681	0.898469	0.898346	0.898334	0.898332
0.910878	0.899617	0.898462	0.898346	0.898334	0.898332
0.910289	0.899556	0.898456	0.898345	0.898334	
0.909727	0.899498	0.89845	0.898344	0.898334	
angle is 60 **the value of k is 1.172873**					
1.009603	1.155207	1.170732	1.17261	1.172841	1.172869
1.018589	1.156061	1.170834	1.172623	1.172842	1.172869
1.027008	1.156872	1.170931	1.172635	1.172844	1.17287
1.034905	1.157644	1.171023	1.172646	1.172845	1.17287
1.042319	1.158378	1.171112	1.172657	1.172847	1.17287
1.049286	1.159076	1.171196	1.172667	1.172848	1.17287
1.05584	1.15974	1.171276	1.172677	1.172849	1.17287
1.062008	1.160372	1.171352	1.172686	1.17285	1.17287
1.067819	1.160973	1.171424	1.172695	1.172851	1.17287
1.073296	1.161545	1.171493	1.172704	1.172852	1.172871
1.078462	1.162089	1.171559	1.172712	1.172853	1.172871
1.083337	1.162607	1.171622	1.17272	1.172854	1.172871

1.08794	1.1631	1.171681	1.172727	1.172855	1.172871
1.092288	1.163569	1.171738	1.172734	1.172856	1.172871
1.096398	1.164015	1.171792	1.17274	1.172857	1.172871
1.100285	1.16444	1.171844	1.172747	1.172858	1.172871
1.103961	1.164844	1.171893	1.172753	1.172858	1.172871
1.107439	1.165229	1.171939	1.172758	1.172859	1.172871
1.110732	1.165595	1.171984	1.172764	1.17286	1.172872
1.11385	1.165943	1.172026	1.172769	1.17286	1.172872
1.116804	1.166275	1.172066	1.172774	1.172861	1.172872
1.119603	1.166591	1.172105	1.172779	1.172862	1.172872
1.122255	1.166891	1.172141	1.172783	1.172862	1.172872
1.12477	1.167177	1.172176	1.172788	1.172863	1.172872
1.127154	1.16745	1.172209	1.172792	1.172863	1.172872
1.129416	1.167709	1.172241	1.172796	1.172864	1.172872
1.131561	1.167956	1.172271	1.172799	1.172864	1.172872
1.133597	1.168191	1.1723	1.172803	1.172865	1.172872
1.135529	1.168414	1.172327	1.172806	1.172865	1.172872
1.137363	1.168627	1.172353	1.172809	1.172865	1.172872
1.139104	1.16883	1.172378	1.172812	1.172866	1.172872
1.140758	1.169023	1.172401	1.172815	1.172866	1.172872
1.142328	1.169207	1.172424	1.172818	1.172866	1.172872
1.14382	1.169382	1.172445	1.172821	1.172867	1.172872
1.145236	1.169548	1.172466	1.172823	1.172867	1.172872
1.146582	1.169707	1.172485	1.172825	1.172867	1.172872
1.147862	1.169858	1.172504	1.172828	1.172868	1.172872
1.149077	1.170002	1.172521	1.17283	1.172868	1.172873
1.150232	1.170139	1.172538	1.172832	1.172868	1.172873
1.15133	1.170269	1.172554	1.172834	1.172868	
1.152374	1.170394	1.172569	1.172836	1.172869	
1.153367	1.170512	1.172584	1.172838	1.172869	
1.15431	1.170624	1.172597	1.172839	1.172869	
angle is 90 the value of k is 0.717865					
0.881153	0.737991	0.719751	0.718035	0.71788	0.717866
0.804387	0.727106	0.718712	0.717941	0.717872	0.717865
0.760577	0.722053	0.718245	0.717899	0.717868	0.717865
angle is 90 the value of k is 1.382319					
1.258321	1.37798	1.382145	1.382312	1.382318	1.382319
1.331381	1.38038	1.382241	1.382315	1.382318	
1.360257	1.381451	1.382284	1.382317	1.382318	
1.372575	1.38193	1.382303	1.382318	1.382319	

angle is 120 the value of k is 0.001629					
0.997046	0.807977	0.618917	0.429872	0.240859	0.051994
0.994092	0.805023	0.615963	0.426918	0.237906	0.049049
0.991137	0.802069	0.613009	0.423964	0.234954	0.046104
0.988183	0.799115	0.610055	0.421011	0.232001	0.04316
0.985229	0.796161	0.607101	0.418057	0.229048	0.040216
0.982275	0.793206	0.604147	0.415104	0.226095	0.037273
0.97932	0.790252	0.601193	0.41215	0.223143	0.034332
0.976366	0.787298	0.598239	0.409196	0.22019	0.031391
0.973412	0.784344	0.595285	0.406243	0.217237	0.028452
0.970458	0.78139	0.592331	0.403289	0.214285	0.025515
0.967503	0.778436	0.589377	0.400336	0.211332	0.02258
0.964549	0.775482	0.586423	0.397382	0.20838	0.019648
0.961595	0.772528	0.58347	0.394428	0.205427	0.01672
0.958641	0.769573	0.580516	0.391475	0.202475	0.013798
0.955686	0.766619	0.577562	0.388521	0.199522	0.010884
0.952732	0.763665	0.574608	0.385568	0.19657	0.007986
0.949778	0.760711	0.571654	0.382614	0.193617	0.005119
0.946824	0.757757	0.5687	0.379661	0.190665	0.002362
0.94387	0.754803	0.565746	0.376707	0.187713	0.001116
0.940915	0.751849	0.562792	0.373754	0.18476	0.002068
0.937961	0.748895	0.559838	0.3708	0.181808	0.001297
0.935007	0.745941	0.556884	0.367847	0.178856	0.001909
0.932053	0.742986	0.55393	0.364893	0.175904	0.00141
0.929098	0.740032	0.550977	0.36194	0.172952	0.001811
0.926144	0.737078	0.548023	0.358986	0.17	0.001484
0.92319	0.734124	0.545069	0.356033	0.167048	0.001749
0.920236	0.73117	0.542115	0.35308	0.164096	0.001533
0.917282	0.728216	0.539161	0.350126	0.161144	0.001708
0.914327	0.725262	0.536207	0.347173	0.158192	0.001565
0.911373	0.722308	0.533253	0.344219	0.15524	0.001681
0.908419	0.719354	0.530299	0.341266	0.152288	0.001586
0.905465	0.7164	0.527346	0.338313	0.149337	0.001663
0.902511	0.713446	0.524392	0.335359	0.146385	0.001601
0.899556	0.710492	0.521438	0.332406	0.143433	0.001652
0.896602	0.707538	0.518484	0.329452	0.140482	0.00161
0.893648	0.704583	0.51553	0.326499	0.13753	0.001644
0.890694	0.701629	0.512576	0.323546	0.134579	0.001616
0.88774	0.698675	0.509623	0.320593	0.131628	0.001639
0.884785	0.695721	0.506669	0.317639	0.128677	0.00162
0.881831	0.692767	0.503715	0.314686	0.125725	0.001635
0.878877	0.689813	0.500761	0.311733	0.122774	0.001623

0.875923	0.686859	0.497807	0.308779	0.119823	0.001633
0.872969	0.683905	0.494854	0.305826	0.116873	0.001625
0.870014	0.680951	0.4919	0.302873	0.113922	0.001632
0.86706	0.677997	0.488946	0.29992	0.110971	0.001626
0.864106	0.675043	0.485992	0.296967	0.108021	0.001631
0.861152	0.672089	0.483039	0.294013	0.10507	0.001627
0.858198	0.669135	0.480085	0.29106	0.10212	0.00163
0.855244	0.666181	0.477131	0.288107	0.09917	0.001628
0.852289	0.663227	0.474177	0.285154	0.09622	0.00163
0.849335	0.660273	0.471223	0.282201	0.09327	0.001628
0.846381	0.657319	0.46827	0.279248	0.09032	0.001629
0.843427	0.654365	0.465316	0.276295	0.08737	0.001628
0.840473	0.651411	0.462362	0.273342	0.084421	0.001629
0.837519	0.648457	0.459409	0.270389	0.081472	0.001628
0.834564	0.645503	0.456455	0.267436	0.078523	0.001629
0.83161	0.642549	0.453501	0.264483	0.075574	0.001628
0.828656	0.639595	0.450547	0.26153	0.072625	0.001629
0.825702	0.636641	0.447594	0.258577	0.069677	0.001629
0.822748	0.633687	0.44464	0.255624	0.066729	
0.819794	0.630733	0.441686	0.252671	0.063781	
0.81684	0.627779	0.438733	0.249718	0.060834	
0.813885	0.624825	0.435779	0.246765	0.057887	
0.810931	0.621871	0.432825	0.243812	0.054941	
angle is 150 **the value of k is 0.733474**					
0.579334	0.746284	0.732431	0.733559	0.733467	0.733475
0.848443	0.724574	0.734204	0.733414	0.733479	0.733474
0.658233	0.739725	0.732964	0.733516	0.733471	0.733474
0.787924	0.729117	0.733831	0.733445	0.733476	
0.696488	0.736527	0.733225	0.733494	0.733472	
0.759797	0.731342	0.733648	0.73346	0.733475	
0.715316	0.734966	0.733352	0.733484	0.733473	
angle is 150 **the value of k is 1.551792**					
1.977378	1.520909	1.554328	1.551585	1.551809	1.551791
1.290575	1.573542	1.55002	1.551937	1.55178	1.551793
1.744472	1.536662	1.553032	1.551691	1.5518	1.551792
1.423739	1.562409	1.550926	1.551863	1.551786	1.551793
1.643884	1.544387	1.552398	1.551743	1.551796	1.551792
1.488853	1.556979	1.551369	1.551827	1.551789	1.551792
1.596437	1.54817	1.552088	1.551768	1.551794	