GROUP 8-16:
NIM TO WIN

An advanced project on the game of Nim

WRITTEN REPORT

Member 1 (Leader): Xu Ziyu 1i3 (30)
Member 2: Jin Zhijia 1i3 (07)
Member 3: Maximilian Ang Jian Zhi 1i3 (19)
Member 4: Rao Zhuoxing 1i3 (24)

Project Work 2019

Table of Contents:

1 Introduction

1.1 Description of Ideas

Nim is a game that can be played by 2 people only. In each game, a random number of sticks will appear in each pile, where a pile refers to a group of sticks. In each turn, a player has to take a number of sticks from one of the piles, but he or she cannot take sticks from multiple piles. The player who picks up the last stick wins. As we found Nim interesting, we wanted to find out more about it.

1.2 Rationale

There is a strategy for the original form of Nim (normal play Nim) but the game is solely dependent on the initial position if both people know the strategy, which makes it boring. Hence, we wanted to develop moving Nim, a variation of normal play Nim, and find the strategy on how to win it. Moving Nim will be introduced later in this report.

1.3 Objective

The objective of our project is to find out the strategies that will allow a player to guarantee a win in normal play Nim and Moving Nim.

1.4 Research Questions

Our 3 Research Questions are:

1. What is the strategy for normal play Nim?
2. What is the strategy for moving Nim, given that there are 3 or less piles and for 3 piles, with a restriction of there are always 7 or less sticks in each pile?
3. How can we prove Research Question 2 Computationally/Mathematically?

1.5 Scope of Study and Terminology

The scope of our study can be summarised with the following

- Combinatorics
- Binary

The following table below shows commonly used terms that we will use throughout this report. (Note that a-b-c is the same as a-c-b, b-a-c, b-c-a, c-a-b, c-b-a)

Terms used	Definition
Losing position	A position where a player starting his/her turn loses and the player that just ended his/her turn wins
Winning position	A position where a player starting his/her turn wins and the player that just ended his/her turn loses
Move	The act of moving a certain amount of sticks from 1 pile to another
Take	The act of removing a certain number of sticks from the same pile
a_1-a_2-...-a_n	Refers to n piles, each containing $a_1, a_2, \ldots, a_n$ sticks respectively

2 Literature Review

Charles L. Bouton gave the game its current name in 1901. He introduced us to the basic rules and gameplay of Nim. Not only that, he managed to develop a full strategy for normal play Nim involving converting numbers into binary form. Furthermore, he also showed how the game is dependent on the initial position as well as whether a player starts first or second no matter what the initial position is.

On the other hand, Arthur Holshouser and Reiter Harold created variations of Nim such as Blocking Nim and Reverse Nim in 2002. Furthermore, they developed the strategies for some of the variations of nim. They also gave us a foundation of possible methods on how to change Nim to make it more interesting and showed various possibilities of new Nim games. However, some of them do not have a solution.

3 Study and Methodology

3.1 Strategies and Methods to develop our Project

To familiarise ourselves with "Moving Nim" several games were needed to fully understand and appreciate what we were dealing with. Not only that extensive reading up was also required before trying to find a strategy because of the newity of this version of Nim. After finding out a strategy, we tried to prove the strategies we had found out.

3.2 Results and Findings

Research Question 1: What is the strategy for normal play Nim?

Firstly, let us introduce a new operation, nim addition ($+_n$), which is the sum of numbers in binary form and when expressed in an algorithm, does not consider any carry over. For example, when we express $3+_n5+_n7$ in an algorithm, we have:

$$\begin{array}{r} 11 \\ 101 \\ +_n \underline{111} \\ 001 \end{array}$$

Hence, $3+_n5+_n7=1$.

We define a current state in game to be "balanced" if the nim sum is equal to 0 and "unbalanced" if otherwise. We discovered that when a player ends his/her turn in a balanced position, he/she is in a winning position.

Take 1-3-5-7 for example. When we express $1+_n3+_n5+_n7$ in an algorithm, we have:

$$\begin{array}{r} 1 \\ 11 \\ 101 \\ +_n\ \underline{111} \\ 000 \end{array}$$

In this case, 1-3-5-7 is a balanced state as it has a nim sum of 0. Hence, starting second will give the player an advantage.

Given that the current position of the sticks is **not** in a balanced state, we must remove sticks such that the nim sum is **zero**. For example, consider 2-4-5-7. When we consider nim sum, we have:

$$\begin{array}{r} 10 \\ 100 \\ 101 \\ +_n\ \underline{111} \\ 100 \end{array}$$

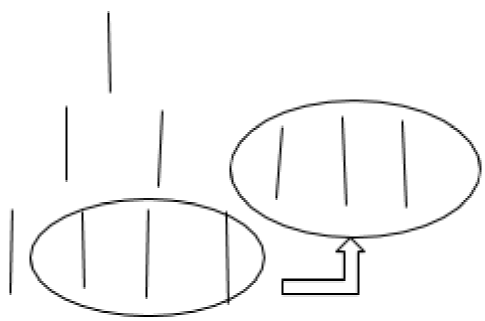
To make the nim sum zero, we need the sum of the first column to be an even number. Hence, we must change 1 of the 3 “1”s in that column to become 0.

For example, we can change the first digit of “101” into 0 such that the binary value becomes 1, which in base-10, is 1. Hence, that player can take away 4 sticks from the pile of 5 to get a balanced state.

$$
\begin{array}{r}
10 \\
100 \\
001 \\
+_n \underline{111} \\
000
\end{array}
$$

Research Question 2: What is the strategy for Moving Nim, given that there are 3 or less piles and for 3 piles, with a restriction of there are always 7 or less sticks in each pile?

Moving Nim is a variation of Nim that we created.



The diagram above shows 3 sticks moved from the third pile to the second pile.

The following are the game rules of moving nim:

- In each turn, a player can either take sticks from the same row or move sticks from 1 row to another, but **cannot** form an extra row.
- Sticks cannot be shifted back to any of the previous positions.
- Moving sticks from 1 pile to another pile such that the number of sticks in those 2 piles interchange is not allowed.

Firstly, let us consider moving nim with only 2 piles.

We noticed that a player should start first if the starting position is not a n-n position or start second if the starting position is in a n-n position.

Example of n-n position	Example of non n-n position
2-2	1-2
3-3	5-13
5-5	29-90

We also realised that the strategy of 2 pile nim is to always try to reduce to a n-n position, which is a losing position as we are going to prove later on.

Secondly, let us consider 3 pile Nim, which have the same rules as 2-pile Nim, but with the additional rule that at any time, there is no more than 7 sticks in a pile.

We noticed that a player should start first if the nim sum of the starting position is not 111 or start second if the nim sum is 111.

Example where player should start first	Example where player should start second
2-3-4 (Nim sum=101)	1-2-4 (Nim sum=111)
3-5-6 (Nim sum=0)	1-3-5 (Nim sum=111)

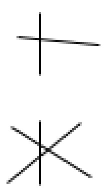
We also realised that the strategy of 3 pile Moving Nim is as follows:

1. Try to reduce a position to a n-n position.
2. If a player is unable to reduce to a n-n position, try to reduce the current position to one with a nim sum of 111.

Research Question 3: How can we prove Research Question 2 Computationally or Mathematically?

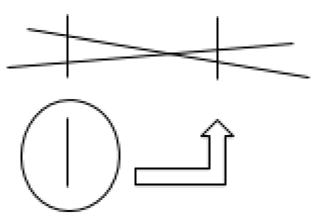
For 2 pile moving nim, we first show that 1-1 is a losing position.
If Player X start first, he can only take 1 stick or move 1 stick to another pile.

If he takes 1 stick, the second player, say Player Y can take the last stick.



The following diagram shows Player X taking a stick with a slash while Player Y takes the remaining sticks with a cross. Player Y wins.

If Player X moves 1 stick from 1 pile to another, Player Y can take both sticks.



Since Player Y, whose moves is represented by a cross, removes the last stick, Player Y wins.

Hence, we can conclude that Player X always loses if he starts first in a 1-1 position.

Now to the general case, starting position n-n.

If move, we have (n-k)-(n+k)	Player Y reduce to (n-k)-(n-k)
If take, we have (n-j)-(n)	Player Y reduce to (n-j)-(n-j)

As shown in the above diagram, Player X can choose to either move or remove the sticks.
In both cases, Player Y is able to make it into a n-n position again.
Thus we can deduce that the n-n position will ultimately reduce to 1-1, which is a losing position for player X.

Now for 3-pile Moving Nim. Firstly we have already proven that n-n is a losing position. Hence we can also conclude that:

- m-n (m≠n) is a winning position
- n-n-m is a winning position
- m-n-m+n is a winning position

Also, take note that a losing position can also refer to a position that when a player takes or moves sticks in any way, it results in a winning position for another player.

Firstly, we found out all the possible positions that have a nim sum of 111.

They are: 1-2-4, 1-3-5, 2-3-6 and 4-5-6

Now we show that each of them is a losing position, starting with 1-2-4.
We group all the possible positions after the player moves or takes sticks from 1-2-4:

Winning position form/Condition	Position after player moves/takes sticks
m-n (m≠n)	1-2, 1-4, 2-4, 1-6, 3-4, 2-5
m-n-m+n	1-2-3
n-n-m	1-2-2, 1-2-1, 1-1-4, 1-1-5, 1-3-3

As all positions after a player move or take sticks from 1-2-4 are winning positions, we have proven that 1-2-4 is a losing position.

It can also be observed that any position that any position that can be reduced to 1-2-4 in 1 turn is a winning position.

We group all the possible positions after the player moves or takes sticks from 1-3-5:

Winning position form/ Condition	Position after player moves/takes sticks
m-n (m≠n)	1-3, 1-5, 3-5, 4-5, 3-6
m-n-m+n	1-3-4, 1-3-2
n-n-m	1-3-3, 1-3-1, 1-1-5, 1-4-4, 1-1-7, 2-2-5, 3-3-3
Reducible to 1-2-4 in 1 turn	1-2-5, 2-3-4

As all positions after a player move or take sticks from 1-3-5 are winning positions, we have proven that 1-3-5 is a losing position.

It can also be observed that any position that any position that can be reduced to 1-3-5 in 1 turn is a winning position.

We group all the possible positions after the player moves or takes sticks from 2-3-6:

Winning position form/ Condition	Position after player moves/takes sticks
m-n (m≠n)	2-3, 2-6, 3-6, 5-6
m-n-m+n	2-3-5, 2-3-1
n-n-m	2-3-3, 2-3-2, 2-2-6, 2-2-7, 3-3-5, 4-3-4
Reducible to 1-2-4 in 1 turn	2-3-4, 2-1-6, 2-4-5, 1-4-6
Reducible to 1-3-5 in 1 turn	1-3-6, 1-3-7

As all positions after a player move or take sticks from 2-3-6 are winning positions, we have proven that 2-3-6 is a losing position.

It can also be observed that any position that any position that can be reduced to 2-3-6 in 1 turn is a winning position.

We group all the possible positions after the player moves or takes sticks from 4-5-6:

Winning position form/ Condition	Position after player moves/takes sticks
m-n (m≠n)	4-5, 4-6, 5-6
m-n-m+n	4-5-1, 4-2-6, 1-5-6
n-n-m	4-5-5, 4-5-4, 4-4-6, 4-4-7, 3-6-6, 5-5-5
Reducible to 1-2-4 in 1 turn	4-5-2, 4-1-6
Reducible to 1-3-5 in 1 turn	4-5-3, 3-5-7
Reducible to 2-3-6 in 1 turn	4-3-6, 3-5-6, 2-5-6, 2-7-6

As all positions after a player move or take sticks from 4-5-6 are winning positions, we have proven that 4-5-6 is a losing position.

It can also be observed that any position that any position that can be reduced to 4-5-6 in 1 turn is a winning position.

In conclusion, we have already proven that 1-2-4, 1-3-5, 2-3-6 and 4-5-6 are losing positions.

Now, we show that it is always possible to reduce a position (that is not a losing position) to a losing position in 1 turn.

Without the loss of generality, we consider a position a-b-c when a≤b≤c.

Firstly, we have already shown that all positions in the form of n-n-m and n-m-m+m can be reduced to a losing position in 1 step.

Now, we show that all positions that is NOT in the form of n-n-m and n-m-n+m or a losing position can be reduced to a losing position in 1 step. We first list all possible such position.

Position	Results: Proven before, when proving a losing position (Proven position is written in cell)?
1-2-n	1-2-4
1-3-n	1-3-5
1-4-n	1-2-4
1-5-n	1-3-5
1-6-n	N/A
2-3-n	2-3-6
2-4-n	1-2-4

2-5-n	N/A
2-6-n	2-3-6
3-4-n	N/A
3-5-n	1-3-5
3-6-n	2-3-6
4-5-n	4-5-6
4-6-n	4-5-6
5-6-7	Can be reduced to 5-6-4 (or 4-5-6)

Now we show that all 1-6-n, 2-5-n and 3-4-n can all be reduced to a losing position in 1 turn.

For 1-6-n, n=7, which results in a m-n-m+n position (losing position)

For 2-5-n, n=6 or 7. Hence, we split into 2 cases:

Case	Result
n=6 (i.e. 2-5-6)	Can be reduced to 2-3-6 (losing position)
n=7 (i.e 2-5-7)	Results in m-n-m+n position (losing position)

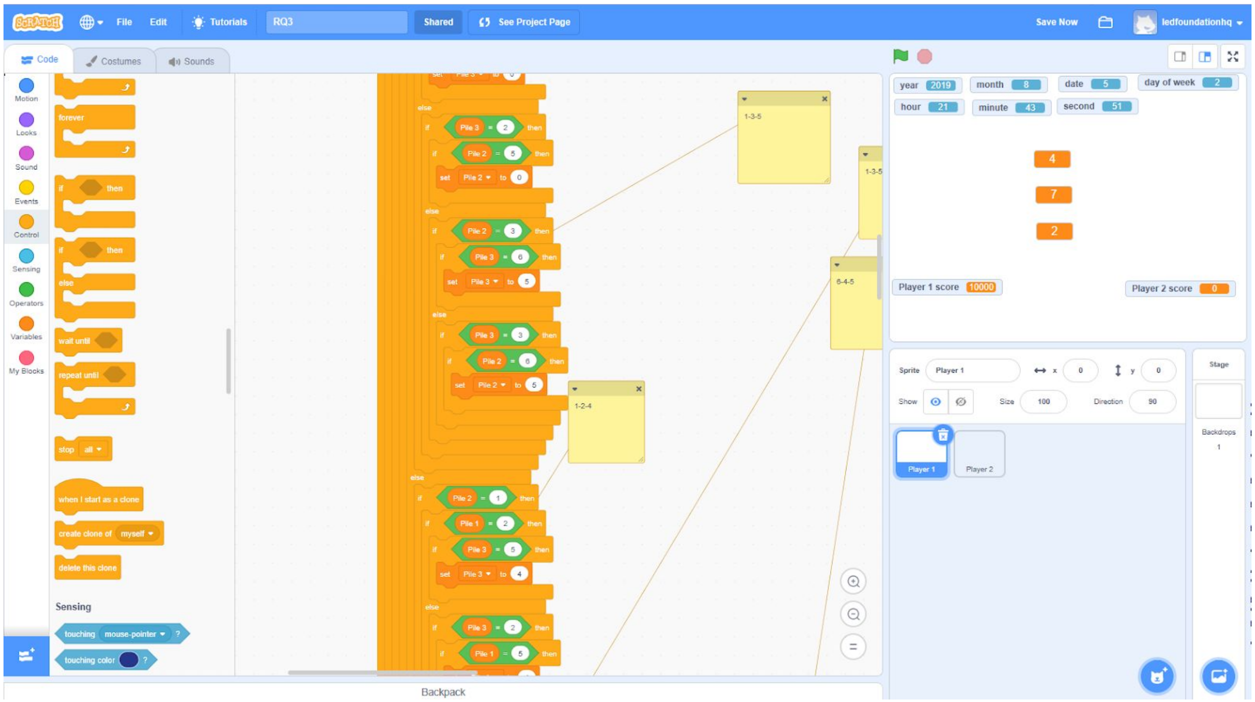
For 3-4-n, n=5, 6 or 7. Hence, we split into 3 cases:

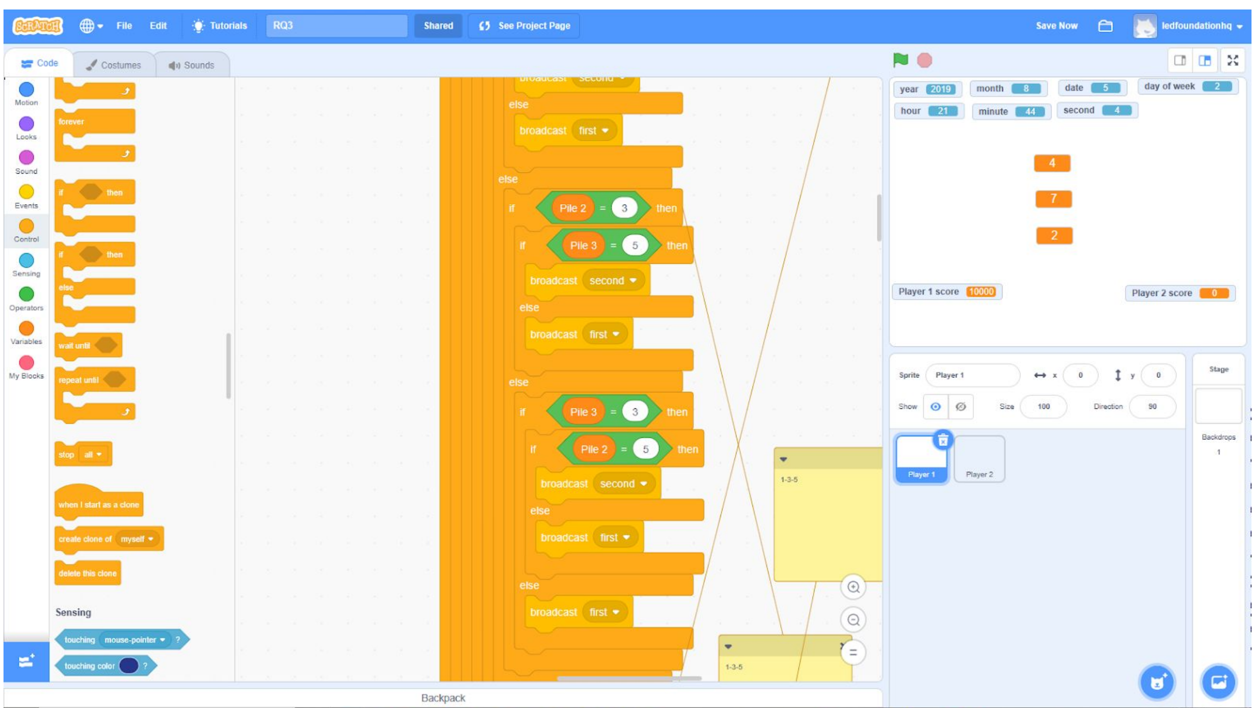
Case	Result
n=5 (i.e. 3-4-5)	Can be reduced to 3-1-5 (or 1-3-5) (losing position)
n=6 (i.e. 3-4-6)	Can be reduced to 3-2-6 (or 2-3-6) (losing position)
n=7 (i.e. 3-4-7)	Results in m-n-m+n position (losing position)

Hence for all 1-6-n, 2-5-n and 3-4-n, it can be reduced to a losing position in 1 turn.

In conclusion, we have proven that all non-losing positions, or winning positions, can be reduced to a losing position in 1 turn. Hence, we have proven that our strategy for 3-pile Moving Nim works.

This can also be proven by code using programming language Scratch. Below is a sample of our code:





This is the final result:

*Note: The 3 big labels represent (from top to bottom) the number of sticks in Pile 1, the number of sticks in Pile 2 and the number of sticks in Pile 3. Since Player 1 won 10000 - 0 and the code was stopped when one of the players reached 10000 points, we have proved that our strategy for 3-pile Moving Nim works.

4 Conclusion

4.1 Outcomes, Analysis and Discussions

In conclusion, we have found out that the strategy for both types of Nim (normal play Nim and Moving Nim) involves either changing a current position to a losing position or nim sum. We also concluded that for both types of Nim, starting first or second affects whether a player can get a guaranteed win.

4.2 Possible Extensions

Having successfully discovered a strategy for Moving nim, given that there are 3 or less piles with seven or less sticks in each pile, and proven it, possible extensions would include coming up with a more general solution, where fewer restrictions are implemented!

5 References

1) Bouton, C. (1901). *Nim, A Game with a Complete Mathematical Theory.* (pp. 35-39). Princeton, New Jersey, United States: Princeton University and the Institute for Advanced Study, Retrieved on:24/05/19

https://www.jstor.org/stable/1967631?origin=crossref&seq=1#metadata_info_tab_contents

2) Arthur Holshouser and Reiter Harold, **Nim with Twists,** 2002, Retrieved on:27/05/19

http://math2.uncc.edu/~hbreiter/NimTwist.