51.84 Degrees

Project 8-05

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Author's Note

THIS is a concise version of the report on all of the authors' research findings throughout the project, intended for submission in the HWA CHONG 2019 PROJECT WORK 7TH CATEGORY. Many results, their respective proofs and their discussions have been omitted for the sake of exercise for the reader. Sections which are less important such as INTRODUCTION AND RATIONALE are shortened, thus motivation for the project may seem less developed and the rigorous foundation may be compromised in favour of more intuitive discussions.

Our work and any material are completely original unless stated explicitly. All irrational values will be in 4 SIGNIFICANT FIGURES and all trigonometric values in 3 SIGNIFICANT VALUES, unless specifically stated.

Nomenclature

β Vertex Angle of Pyramid

θ Slope angle of Pyramid

a Apothem of Pyramid

d Distance from Base of Pyramid to Centre of Gravity

h Height of the Pyramid

l Base Length of Pyramid

S Surface Area of a Triangular Face of Pyramid

Contents

1 Introduction and Rationale

51.84°is half of the vertex angle of a cone that will be created when dry sand is poured from a spot at 0 degrees, no matter how much sand is there, if the land is flat. This is a very interesting phenomenon because the angle of the side angle of the Great Pyramid of Khufu *(fig.1)* is also 51.84°.

We wonder whether it is a coincidence that the angles in both examples are the same or whether there are some mathematics behind them. Therefore, we embarked on this project to discover more about it.

Figure 1: The Great Pyramid of Giza

2 Objectives

i To find the relationship between the value of this angle and other special numbers such as The Golden Ratio, Pi etc.

ii To validate that the angle of the Great Pyramid has to be 51.84°.

iii To show that 51.84°is the optimal angle for the Pyramid of Khufu.

iv To find the most stable regular n-sided polygon-based pyramid with the slope angle of 51.84°.

2.1 Research Questions

i What is the relationship between the value of this angle and other special numbers such as The Golden Ratio, Pi etc?

ii How do we validate that the Great Pyramid has to be the angle of 51.84°?

iii How do we show that 51.84°is the optimal angle for the Pyramid of Khufu?

iv What is the value of n?

3 Literature Review

3.1 the *seked* system

Definition 3.1. the *seked* system
The seked was an ancient Egyptian unit for the measurement of the slope of an inclined surface. The system was based on the Egyptian's linear measure known as the royal cubit. The royal cubit was subdivided into seven palms and each palm was further divided into four digits. The inclination of measured slopes was therefore expressed as the number of palms and digits moved horizontally for each royal cubit rise.

Information on the use of the seked in the design of pyramids has been obtained from two mathematical papyri; the Rhind Mathematical Papyrus in the British Museum and the Moscow Mathematical Papyrus in the Museum of Fine Arts. The Rhind Mathematical Papyrus, particularly, stated in question 61:

> If a pyramid is 250 cubits high and the side of its base 360 cubits long, what is its seked?

The solution to the problem is given as the ratio of half the side of the base of the pyramid to its height, or the run-to-rise ratio of its face. In other words, the quantity found for the seked is the cotangent of the angle to the base of the pyramid and its face *(Maor, 1998).*

The most famous of all the pyramids of Egypt is the Great Pyramid of Giza built around 2,550 B.C.. Based on the surveys of this structure that have been carried out by Flinders Petrie and others, the slopes of the faces of this monument were a seked of 5 1/2, or 5 palms and 2 digits, which equates to a slope of 51.84°from the horizontal, using the modern 360 degree system *(Petrie, 1893).*

3.2 Dimensions of the Great Pyramid

3.2.1 Base Measurements of the Great Pyramid

THE Pyramid of Khufu's measurements has been debatable for decades, due to the level of erosion that happened to the outer layer of the Pyramid. Yet, part of the Pyramid's initial outer layer still exists at the apex *(refer to fig. 1)* of the Pyramid, making the initial construction measurements still be able to be plotted out, mostly due to the advancements in technology in today's society. These measurements will be very important to our research, as we will be analysing the symmetry and correlation between these measurements in OBJECTIVE 1 AND OBJECTIVE 2 .

3.2.2 Method of Measurement

Many surveyors, including most notably Sir. Flinders Petrie and J. H. Cole, has used different methods to measure the dimensions of the Great Pyramid, particularly the base length and height. These different methods also yielded different results. For this research, we will be using the survey "Determination of the Exact Size and Orientation of the Great Pyramid of Giza" by J. H. Cole in 1925 due to this survey being widely accepted in the mathematical society.

The survey particularly states the method of measurement:

> Eight brass bolts were cemented into the rock round the base, one near each of the four corners, and, as these were not intervisible, four more were placed, one at about the middle of each side, in such a position that each point was visible from the adjacent points on each side of it.
>
> These bolts are numbered from 1 to 8 clockwise from No.1 which is at the S.E. corner of the Pyramid. *(Cole, 1925)*

Remark. The following data will be in their respective forms of accuracy as appeared in the original survey.

3.2.3 Base Length of the Pyramid

Cole reported the following data for the length of the sides:

North	230,523 mm.
East	230,391 mm.
South	230,454 mm.
West	230,357 mm.
Mean	230,364 mm.

Table 1: *Base Length of the Great Pyramid*

Cole calculated also the maximum possible error, due to the state of the remains, of the difficulty to determine corners with absolute exactitude:

North	6 mm. at either end
East	6 mm. at either end
South	10 mm. at West end, 30 mm. at East end
West	30 mm. at either end

Table 2: *Maximum Error of Base Length Measurements*

Thus it was agreed that the Great Pyramid of Khufu had sides of 440 Egyptian royal cubits.

Remark. 1 Egyptian royal cubit is equivalent to 524 mm.

3.2.4 Height and Slope Measurements of the Pyramid

At the present state of the remains, it has proved impossible to calculate the slope with mathematical exactness. Petrie performed several tests on the North side, which is best preserved, and arrived at an average of 51°50′40″. But in one test on the South side he obtained 51°57′30″ ; this caused him to raise the question whether the slope was different on different faces.

Because of his doubt concerning the identity of the slope on the four faces, Petrie measured the terrace which exists today on the top, and found that the North edge is more distant from the vertical axis of the Pyramid than the three other sides; but he himself observed that at present the terrace does not include the casing.

Petrie was unduly impressed by the unevenness of the terrace and somehow took the aberrant single datum for the South side into account; he concludes:

> On the whole, we probably... cannot do better than take 51°52′ ± 2′ as the nearest approximation to the mean angles of the Pyramid, allowing some weight to the South side.

Cole in his survey assumed that the height relates as 7:51/2 to the half basis and found this relation fitting to his observations. He found that some of the casing blocks at the very foot of the Pyramid are still in place and are well preserved for a substantial length on the North side, with a well recognisable angle of the facing. From this and from other data he concluded that the slope is 51°50′40” ± 1′05”.

Petrie concludes that the Pyramid had a height of 146.71 m. ± 0.18. In my opinion the Pyramid had a height of 280 cubits, so that, by computing the cubit as 524 mm., the height is 146.72 m.

3.3 Vertex Angle of Sand

Using natural materials such as sand, smooth stone and flour, pouring the material down at a flat surface will creating a cone that has the vertex angle of 51.84°*(李剑桥, 竭宝峰, 2008).*

4 Research Findings

4.1 Objective 1

> *To find the relationship between the value of this angle and other special numbers such as The Golden Ratio, Pi etc.*

4.1.1 The Golden Ratio

In relation to the direct measurements of the Great Pyramid, we found 3 relations between the Great Pyramid and ϕ.

i If you divide the surface of the base l^2 by the of the rest of the lateral surface $4S$, you will get the major of the golden ratio:

$$\frac{l^2}{4S} = maj.\theta = \phi - 1$$

ii The cosine of the angle of the Great Pyramid θ, will result in the major of the golden ratio:

$$\cos\theta = maj.\theta = \phi - 1$$

iii The apothem of the Great Pyramid a and half its base length $\frac{1}{2}l$ is the golden ratio:

$$\frac{a}{\frac{1}{2}h} = \frac{356ERC}{220ERC} = 1.618 \approx \phi$$

4.1.2 Pi

We also found 2 relations between Pi and the Great Pyramid:

1. Divide the Great Pyramid's perimeter $4l$ by its height h and π is obtained:

$$\frac{4l}{h} = \frac{1760ERC}{280ERC} \approx 3.142$$

2. The slope of the Great Pyramid $\tan\theta$ is equivalent to $\frac{4}{\pi}$:

$$\tan\theta = \frac{4}{\pi} = 1.273$$

4.1.3 Euler's Number

We also found out that the constant e, known as Euler's number, is the result of $4 \times$ the slope angle θ divide by the vertex angle β.

$$4 \times \frac{\theta}{\beta} = e$$

4.2 Objective 2

To validate that the angle of the Great Pyramid has to be 51.84°.

4.2.1 The Golden Ratio / Pythagoras' Theorem

We found out that a pyramid based on Phi varies by only 0.025% from the Great Pyramid's estimated dimensions. Phi is the only number which has the mathematical property of its square being one more than itself.

$$\phi + 1 = \phi^2$$

or

$$1.618 + 1 = 2.618$$

Theorem 1. *The Pythagoras' Theorem*

In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

$$a^2 + b^2 = c^2$$

By applying the above Pythagorean equation to this, we can construct a right triangle, of sides a, b and c, or in this case a Golden Triangle of sides $\sqrt{\phi}$, 1 and ϕ *(fig. 2).*

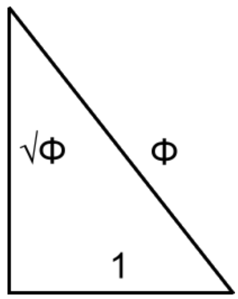

Figure 2: Triangle formed using Golden Ratio Proportions

This creates a pyramid with a base width of 2 (i.e., two triangles above placed back-to-back) and a height of the square root of Phi, 1.272. The ratio of the height to the base is 0.636. Translating this into math terms:

$$h^2 = \frac{la}{2}$$

or

$$h = \sqrt{\frac{la}{2}} \tag{1}$$

Substituting Height, Base Length and Apothem of the Great Pyramid:

$$146.721 = \sqrt{\frac{(230.363)(186.531)}{2}} = 146.577$$

The above numbers are not very different (absolute difference: 0.144, relative error: 0.098%), but they does not show any relationship with the Golden Ratio. To see this relation it is necessary to do some algebra, and use the Pythagorean Theorem:

$$h^2 = \frac{la}{2} = \frac{l}{2}a$$

$$h^2 = \frac{l}{2} \times \sqrt{\left(\frac{l}{2}\right)^2 + h^2}$$

$$\frac{h^4}{\left(\frac{l}{2}\right)^2} = \left(\frac{l}{2}\right)^2 + h^2$$

$$\frac{h^4}{\left(\frac{l}{2}\right)^2} = 1 + \frac{h^2}{\left(\frac{l}{2}\right)^2}$$

$$\left(\frac{2h}{l}\right)^4 = \left(\frac{2h}{l}\right)^2 + 1$$

Taking $u = \left(\frac{2h}{l}\right)^2$:

$$u^2 = u + 1$$

Solving this equation:

$$u = \left(\frac{2h}{l}\right)^2 = \frac{1 \pm \sqrt{5}}{2}$$

Discarding the *minus* solution because this is negative, then the solution of this equation is the *Golden Ratio*:

$$\left(\frac{2h}{l}\right)^2 = \frac{1 \pm \sqrt{5}}{2} = \phi \tag{2}$$

So the squared ratio of twice the Great Pyramid's height to its base side length, should be the *Golden Ratio*. Verifying this using Height and Base Length of the Great Pyramid:

$$\left(\frac{2 \times 146.721}{230.363}\right)^2 = 1.62263$$

Compared with exact *Golden Ratio* value:

$$\phi = \frac{1 + \sqrt{5}}{2} = 1.61803$$

the difference between these two values is

$$1.62263 - \phi = 0.000459$$

and the relative error from ϕ is:

$$\frac{1.62263 - \phi}{\phi} \times 100 = 0.28396\%$$

This can be also considered a fair match, but this error is 17 times greater than the construction error.

Remark. We approximate the value of The Golden Ratio at *1.618*

4.2.2 Pi

ONE reasonable geometric criteria that the ancient Egyptians maybe used to define the proportions of the Great Pyramid is through the use of Pi. It has been found that the perimeter of the Pyramid's base is equal to the length of a circumference with radius equal to the Pyramid's height.

Translated into Math terms and using the Height of the Great Pyramid and the Base Length of the Great Pyramid:

$$4l \approx 2\pi h$$

$$\text{or}$$
$$\frac{2l}{h} \approx \pi \tag{3}$$

Using the Height and Base Length of the Great Pyramid:

$$\frac{2 \times 230.363}{146.721} = 3.14015 \tag{4}$$

The calculated difference and relative error from π, rounding to 5 decimals are:

$$3.14015 - \pi = -0.00144$$

$$\frac{3.14015 - \pi}{\pi} \times 100 = -0.04592\%$$

4.3 Objective 3

To show that 51.84°is the optimal angle for the Pyramid of Khufu

4.3.1 Slope angle affects height of Pyramid

ASSUMING the base length l is constant, the height of the Pyramid is the result of half the base length of the Pyramid, multiplied by the tangent of θ:

$$h = \frac{1}{2}l(\tan\theta) \tag{5}$$

We plotted the data into a graph shown below *(fig. 3)*:

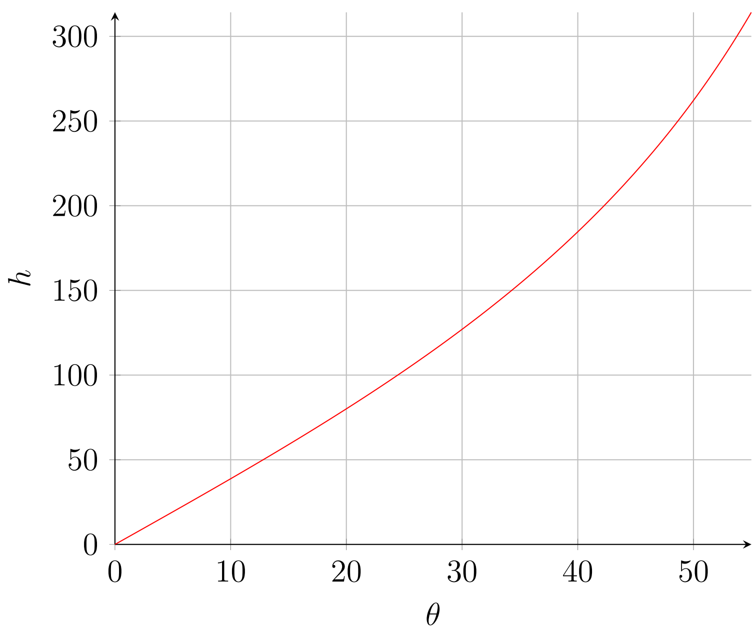

Figure 3: Graph of formula $h = \frac{1}{2}l(\tan\theta)$

We can conclude from the graph that as the slope angle θ of a pyramid increases, its height h also increases.

4.3.2 Base length affects volume of Pyramid

ASSUMING constant slope angle θ is constant, using the formula to calculate the volume of pyramid:

$$Volume = \frac{l^2 h}{3}$$

Substituting equation (5) into the formula:

$$Volume = \frac{l^2 \times \frac{1}{2} l(\tan(\theta))}{3}$$

Simplifying the equation:

$$Volume = \frac{l^3 \times (\tan(\theta))}{6} \tag{6}$$

We plotted equation (6) into a graph *(fig. 4)*:

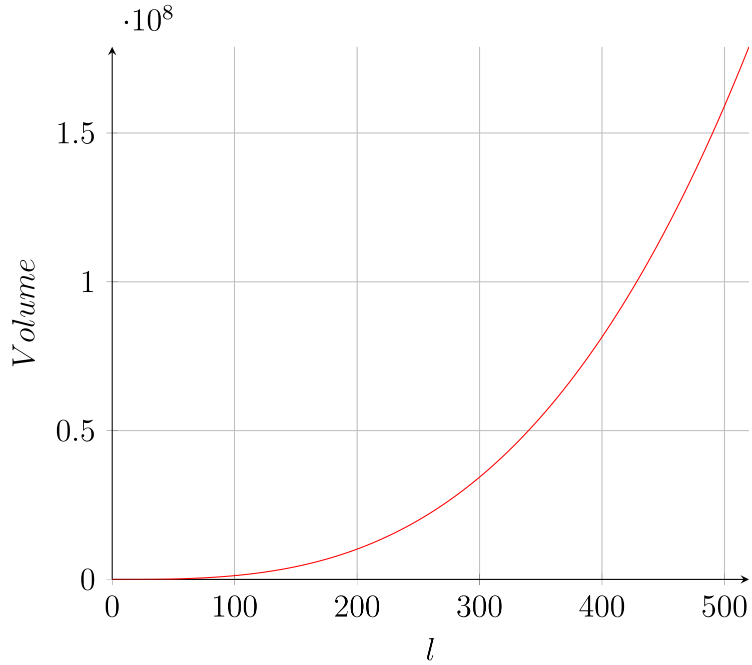

Figure 4: Graph of formula $Volume = \frac{l^3 \times (\tan(\theta))}{6}$

From the graph, we can imply that as the base length of a pyramid increases, the volume of the pyramid increases.

4.3.3 Height affects distance from base of pyramid to its centre of gravity

Theorem 2. *The geometric centre of gravity for the pyramid is $\frac{1}{4}$ the height of the pyramid.*

$$d = \frac{h}{4}$$

Assuming a constant base area, the centre of gravity for a pyramid can be obtained using THEOREM 2.

Plotting the equation into the graph *(fig. 5)*:

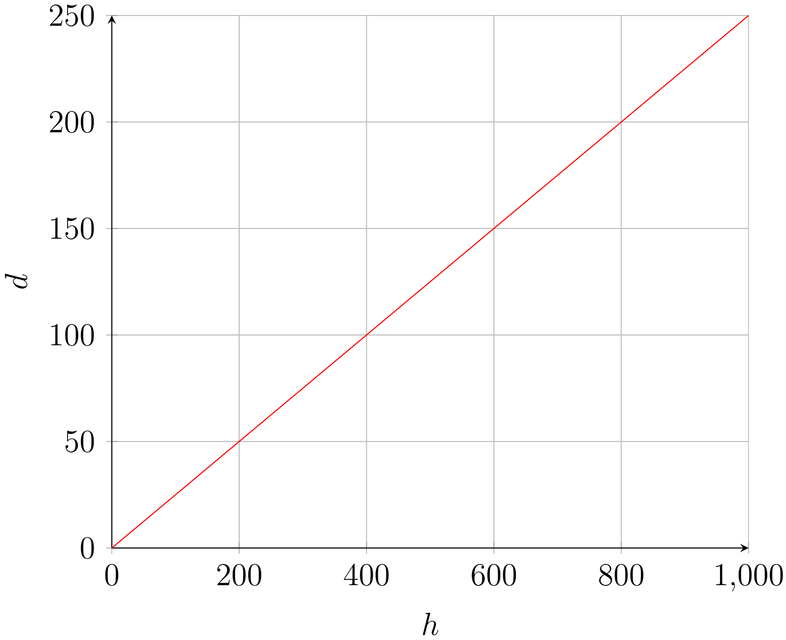

Figure 5: Graph of formula $d = \frac{h}{4}$

We can infer from this graph that as the height of the pyramid increases, the distance of the base to the centre of gravity of the pyramid increases.

4.3.4 Slope angle affects volume of a pyramid

Using equation (6), we can plot volume of the pyramid against its slope angle, with the base length of the pyramid l as a parameter to vary the different curves. *(fig. 6)*:

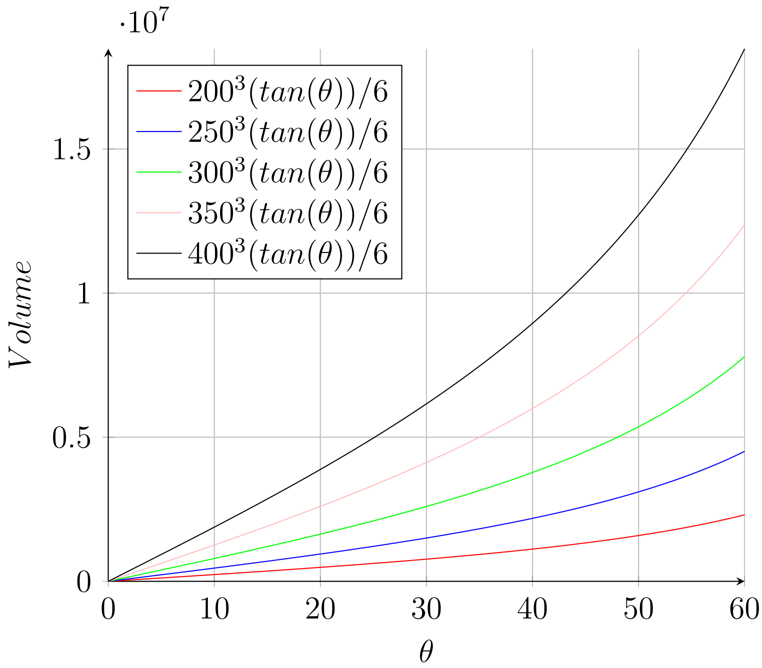

Figure 6: Graph of formula $\frac{l^3(\tan(\theta))}{6}$

From the graph, we can see that when the slope angle increases, the volume increases. When the base length l increases, the volume increases.

4.4 Objective 4

To find the most stable regular n-sided polygon-based pyramid with the slope angle of 51.84°.

4.4.1 General Formulae

To calculate the base area of the pyramid A:

$$A = \frac{n}{4} l^2 \cot\left(\frac{180^\circ}{n}\right) \tag{7}$$

To calculate the volume of the pyramid:

$$Volume = \frac{Ah}{3} \tag{8}$$

To calculate the apothem of the n-sided polygon:

$$Apothem = \frac{l}{2\tan\left(\frac{180^\circ}{n}\right)} \tag{9}$$

4.4.2 Calculations

To obtain the height of a pyramid h, we use the trigonometric function tan():

$$h = Apothem \times \tan 51.84$$

Substitute equation (9) into this equation:

$$h = \frac{l \tan 51.84°}{2 \tan\left(\frac{180°}{n}\right)} \tag{10}$$

Substituting equation (7) and (10) into equation (8):

$$Volume = \frac{\frac{n}{4} l^2 \cot\left(\frac{180°}{n}\right) \times l \tan 51.84°}{2 \tan\left(\frac{180°}{n}\right)} \times \frac{1}{3}$$

Simplifying the equation:

$$Volume = \frac{\frac{n}{4} l^3 \cot\left(\frac{180°}{n}\right) \times l \tan 51.84°}{6 \tan\left(\frac{180°}{n}\right)} \tag{11}$$

4.4.3 Results

Substitute equation (7) into equation (10):

$$Volume = \frac{n}{4} l^2 \cot\left(\frac{180°}{n}\right) \times \frac{h}{3} \tag{12}$$

We plot a graph using the equations (12) and (10) *(fig. 7)*:

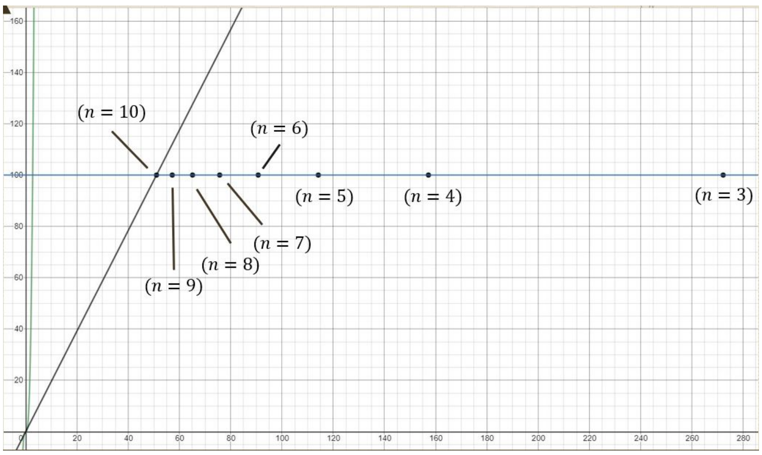

Figure 7: Graph of height against volume

This figure shows that when n increases, the height of the pyramid decreases. As all the pyramids have a constant volume, when the height of pyramid decreases, the base area increases. For the pyramid to be most stable, n should be infinity. Therefore, the base of the pyramid is a circle.

5 Conclusion

1. The angle of 51.84°is related to **Phi** and **Pi** as shown in the Great Pyramid.
2. The principle of **Pi** as well as **Pythagoras Theorem** can be used to create a pyramid similar to the Great Pyramid.

3. We managed to find the formulae that relates the volume of the pyramid to its base length and slope angle. However, the volume and base area could not be plotted on the same axis due to the differences in their units. Thus, Objective 3 is partially completed.

4. The cone is the most stable regular n-sided polygon-based pyramid with the slope angle of 51.84°.

6 Limitations and Further Extensions

We could not find a clear relationship between 51.84°and its volume to base area ratio. Thus our further extension is to find a clear relationship between the distance from base to centre of gravity and the base area through graphs to prove that 51.84°is the optimal angle for Pyramid of Khufu.

7 Acknowledgement

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