The Lucky Cat

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1. Introduction

1.1 Abstract

Our project applies logic to find a cat hiding in certain number of boxes, with a special rule restricting the movement of the cat. We need to open box(es) step by step to reduce the number of boxes that the cat may hide inside and thus find it. By analyzing the effectiveness of opening different boxes, we can find a simplest way to find the cat. There is also a formula to represent the relationship between the number of boxes and the least steps needed to find the cat in different cases.

1.2 Introduction of the rule

A cat is hiding in one of the n (For example, n is 5) boxes lined up in a row (Figure 1). You can open one box at a time to check whether the cat is inside it or not. If not, you should close the box immediately and keep your eyes off the five boxes. The cat will quickly move into the box next to the previous one. After the cat changes its position, you can open one box again and repeat this procedure until you find the cat. However, the cat is so lucky that it always hides in a box that we did not open.

1.3 Literature review

According to relevant information online, we found that predecessors have come up with a method that can definitely find the cat. If there are n boxes, then the least number of steps of opening a box to find the cat is $2(n-2)$, but it has not been proved to be the simplest way. Moreover, it is interesting to change the number of boxes and the arrangement of them (such as a closed graph) since we suppose these variables will probably influence the results. By investigating these, we can understand the question in a deeper manner and develop our ability of analyzing and applying logics.

Figure 1

1.4 Objective

- To further explore the existing method of finding the cat with a fixed order.
- To investigate the different variables that can be introduced into the question.

1.5 Research questions

- How can we find the cat hiding in one of the *n* identical boxes in a row by the simplest way and how to prove it?
- How will the the number of boxes that can be opened each time impact the result?
- How will the arrangement of the boxes (a closed graph) impact the result?

The solution and result of these questions will be discussed in part 2 Methodology.

2. Methodology

2.1 Procedure

We used a chart (Figure 2) to discuss the possible boxes that the cat may hide inside. The columns represent the boxes and the rows represent the steps of opening the box.

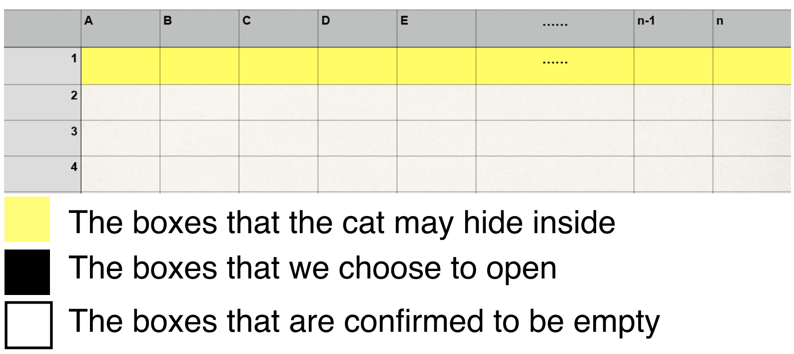

Figure 2

Firstly, we find that there is no effect to confirm two adjacent boxes (Figure 3.1) to be empty as for the next step, the cat may still move into them.

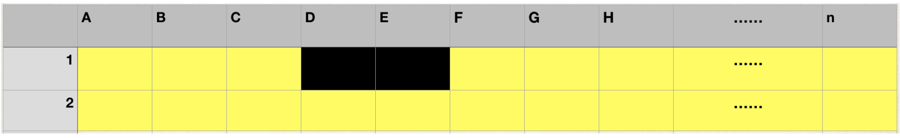

Figure 3.1

Moreover, it is also useless to confirm two boxes to be empty, leaving more than one box in between (Figure 3.2) for the same reason.

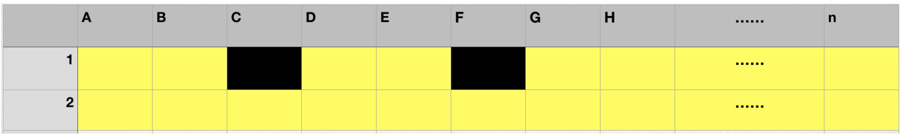

Figure 3.2

Only when there is just one box between two boxes which are confirmed to be empty (Figure 3.3), we can get new results as the cat cannot hide in this box in the next step.

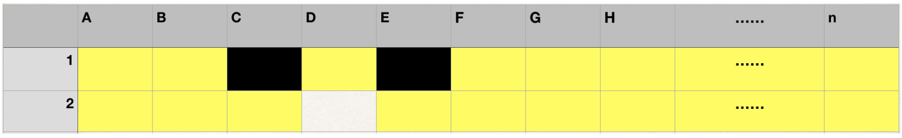

Figure 3.3

With this understanding, we can start to solve the problems.

2.2 Research question1

How can we find the cat hiding in one of the *n* boxes in a row in the simplest way and how to prove it?

For the first step, only opening the second (Box B) or the second last (Box *n-1*) box is useful since there is no effect by opening other boxes (Figure 4.1).

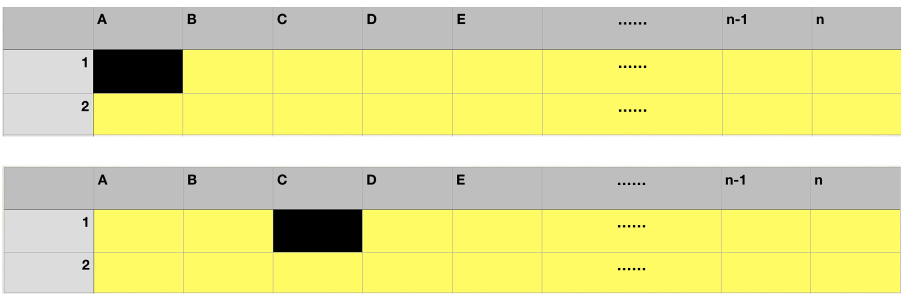

Figure 4.1

If we open the second box (Box B) in the first step, the cat cannot hide in Box A in the next step (Figure 4.2).

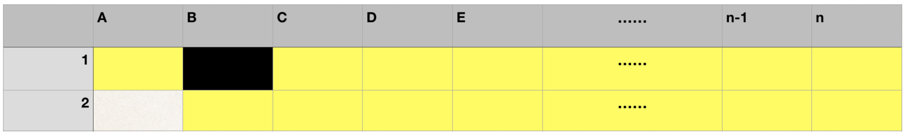

Figure 4.2

As in the second step, Box A is confirmed to be empty. If we choose to open Box C, we can confirm Box B to be empty in the next step, or we can choose to open Box *n-1* as it can confirm Box *n* to be empty in the next step. Otherwise, there is no effect since the cat may still hide in any of the boxes.

If opening Box *n-1*, it is not necessary to open Box B in the first step as the result is the same (Figure 5.1).

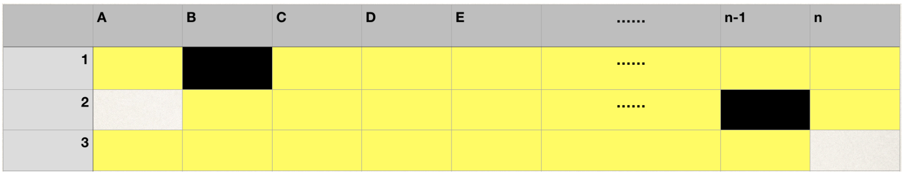

Figure 5.1

Therefore, we should open Box C in the second step (Figure 5.2).

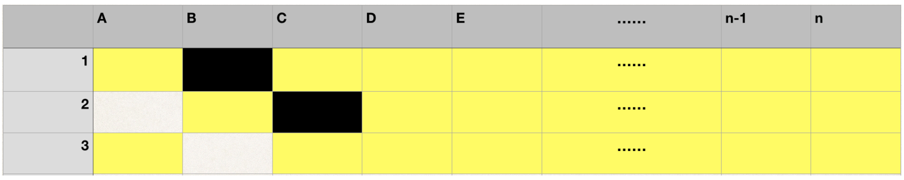

Figure 5.2

Similarly, we should open Box D in the third step to make sure Box C is empty in the next step, and the empty Box B itself can also confirm that Box A is empty in the next step (Figure 6).

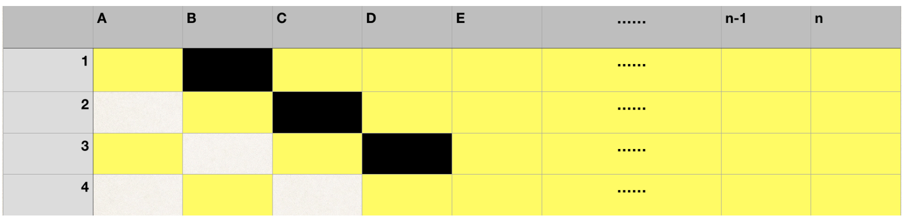

Figure 6

However, in the fourth step, both opening Box B and Box E can result in a new result. If we open Box B, we can confirm Box A to be empty in the next step (Figure 7.1).

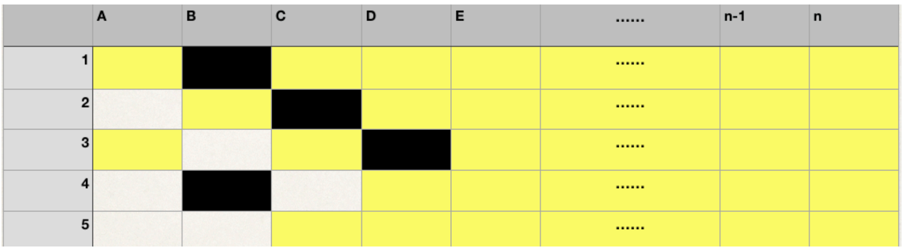

Figure7.1

However, whatever we choose to open in the fifth step, there is no effect as the result has occurred before. Therefore we need to open Box E in the fourth step (Figure 7.2).

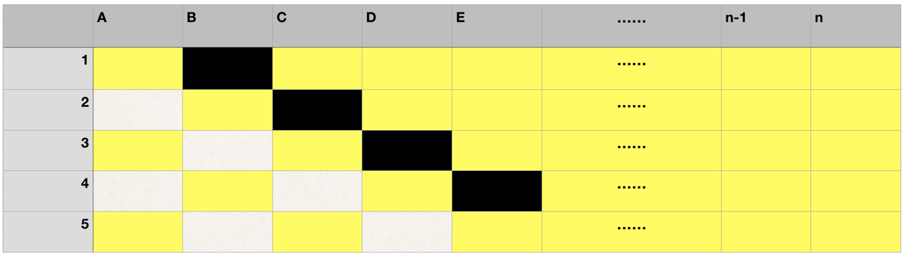

Figure7.2

Following the pattern, we open the boxes from the second one to the second last one which needs *n-2* steps. Then all the boxes that the cat may hide inside are separated by a box which is confirmed to be empty.

When the number of boxes *n* is odd (Figure 8.1):

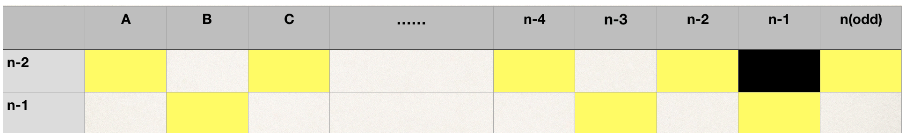

Figure 8.1

Only opening either Box B or Box *n-1* in the *(n-1)th* step is effective as opening other boxes will lead to the same result as in the *(n-2)th* step (Figure 8.2).

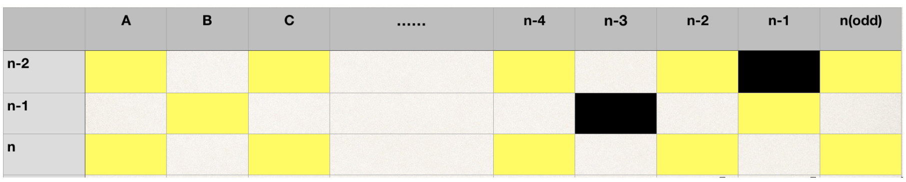

Figure 8.2

Take opening Box *n-1* as an example, we can confirm that Box *n* is empty in the next step (Figure 8.3).

Figure 8.3

Then we should open Box *n-2* as this can make sure Box *n-1* is empty in the next step (Figure 8.4).

Figure 8.4

Similarly, we need to open the boxes from the second last box to the second box as at last the cat can only be in Box B. The least steps needed is *2n-4* (Figure 8.5).

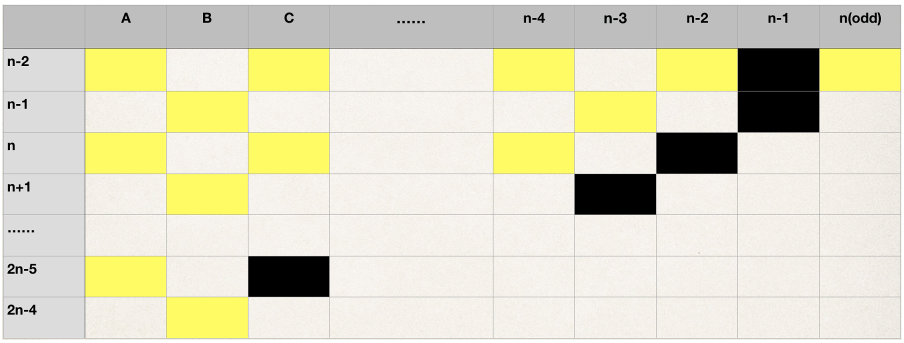

Figure 8.5

When the number of boxes *n* is even (Figure 9.1):

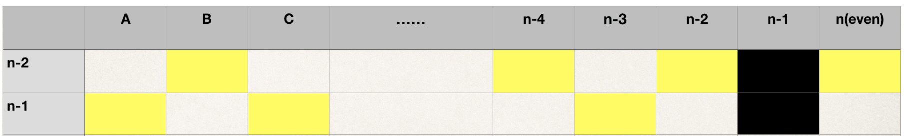

Figure 9.1

Similarly, we should open from Box *n-1* to Box B one by one, as there is an extra step to open from the other side which means to open from Box A to Box *n-1*. It is because there is no effect by opening Box A in the *(n-1)th* step.

Finally, it also requires at least *2n-4* steps to find the cat (Figure 9.2).

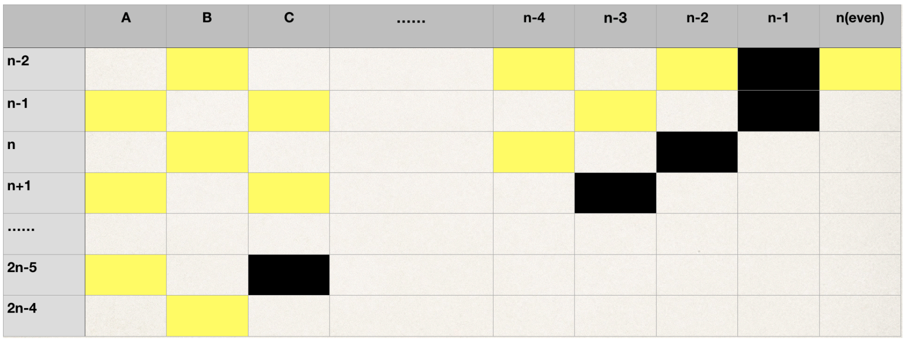

Figure 9.2

2.2 Research question 2

How will the the number of boxes that can be opened each time impact the result?

Firstly, if we are allowed to open two boxes per time, it has been proved before that we need to open two boxes which are separated by one box (Figure 10.1), or there is no effect by opening two boxes.

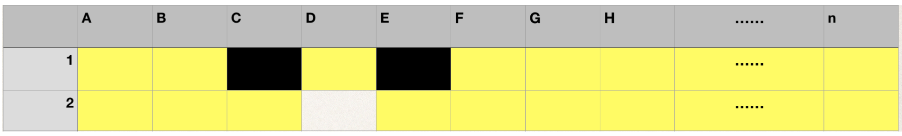

Figure 10.1

If one of the two boxes we choose to open is Box B or Box *n-1*, then we can confirm two boxes to be empty which is more effective (Figure 10.2).

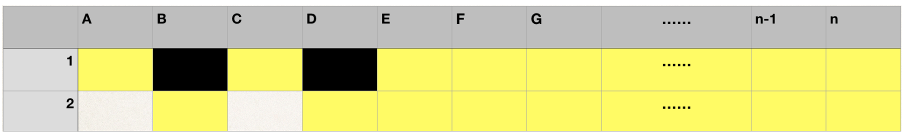

Figure 10.2

In order to make use of every empty box to save steps, similar to the first question, we need to separate the boxes that the cat may hide inside (Figure 11).

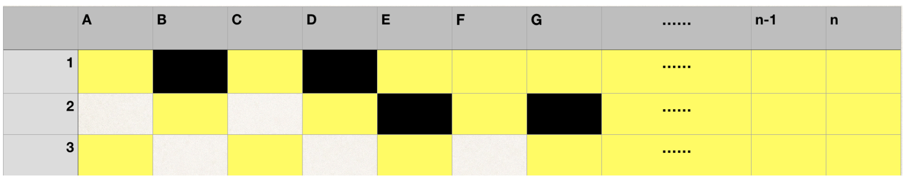

Figure 11

After separating all the boxes, there are three possible patterns of the boxes the cat may hide inside (Figure 12).

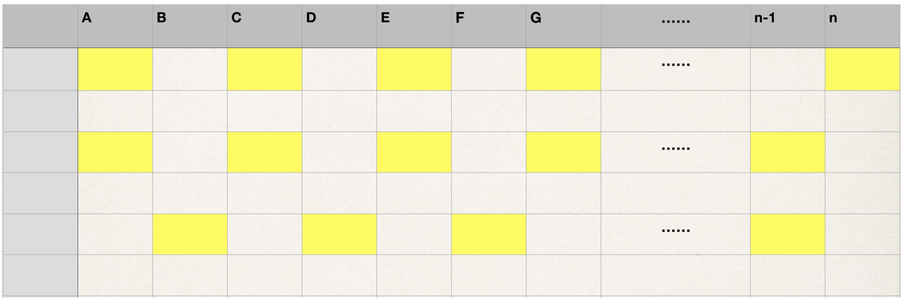

Figure 12

In the first pattern (Figure 13.1), we still need to confirm two boxes with another empty one in between to be empty (Figure 13.2).

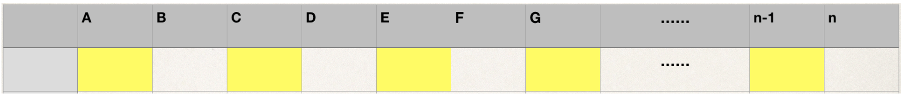

Figure 13.1

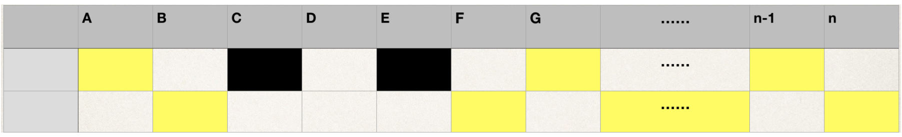

Figure 13.2

If we do not choose Box B and Box F in the second step, the cat may move to Box C and Box E again, making the first step useless (Figure 13.3).

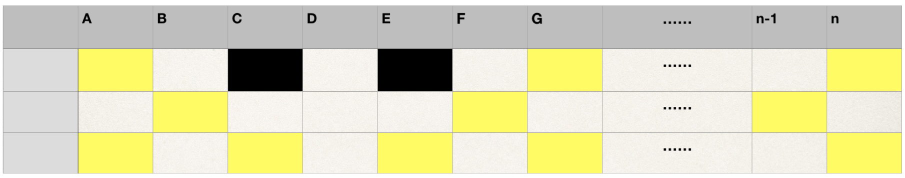

Figure 13.3

Therefore, we need to open the boxes from the boxes opened in the first step to both two sides. If one side of boxes are all confirmed to be empty (For example in the Figure 13.2, boxes on the left side are empty), then we need to continue to open the other side of boxes from the starting side to the end (Figure 13.4).

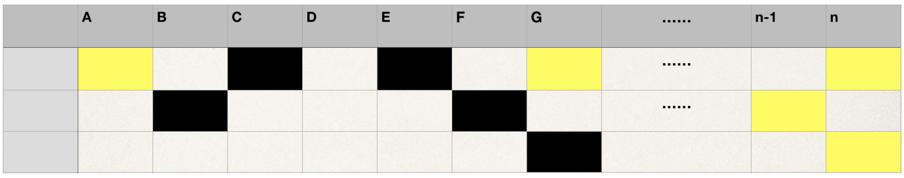

Figure 13.4

Another way is to directly open the boxes from one side to the other (Figure 13.5) and it requires the same number of steps.

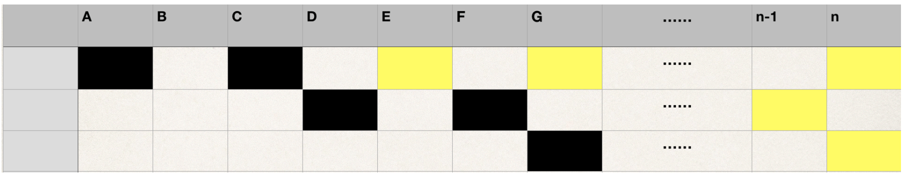

Figure 13.5

In the second pattern (Figure 14.1), we should open the boxes from the side where the cat may hide inside the second box to the other (Figure 14.2) as we can also make sure the first box is empty in the next step.

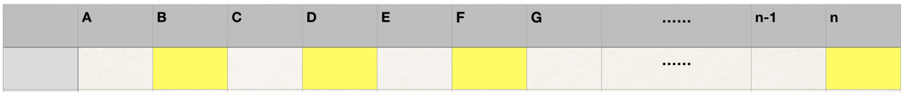

Figure 14.1

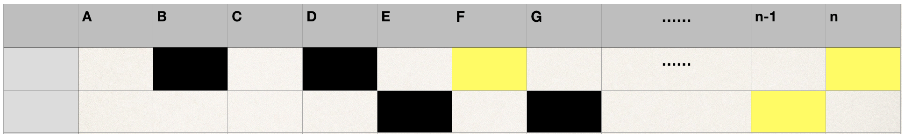

Figure 14.2

In the third pattern (Figure 15.1), there are two ways to open the boxes.

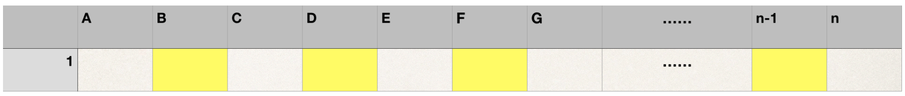

Figure 15.1

First is to open from one side to the other (Figure 15.2) and this requires $\lfloor n/3 \rfloor$ steps.

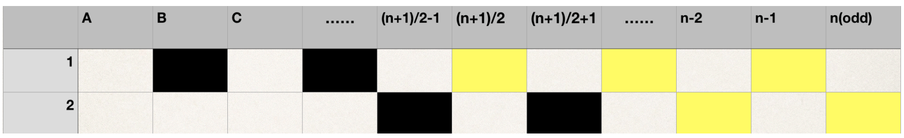

Figure 15.2

The other one is to open from both sides to the middle as the second and the second last box can confirm the first and the last box to be empty in the next step and it requires *(n-1)/2* steps (Figure 15.3).

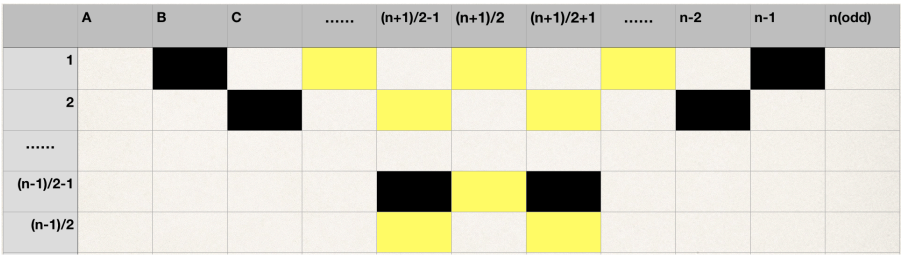

Figure 15.3

After plotting the graph of x(number of steps needed) against n(number of boxes) for both cases, we find that in the range of n ($n \geqq 3$), the value of $\lfloor n/3 \rfloor$ is always smaller than *(n-1)/2* (Figure 16.1). Therefore we can conclude that choosing from one side to the other is always the simplest way to find the cat.

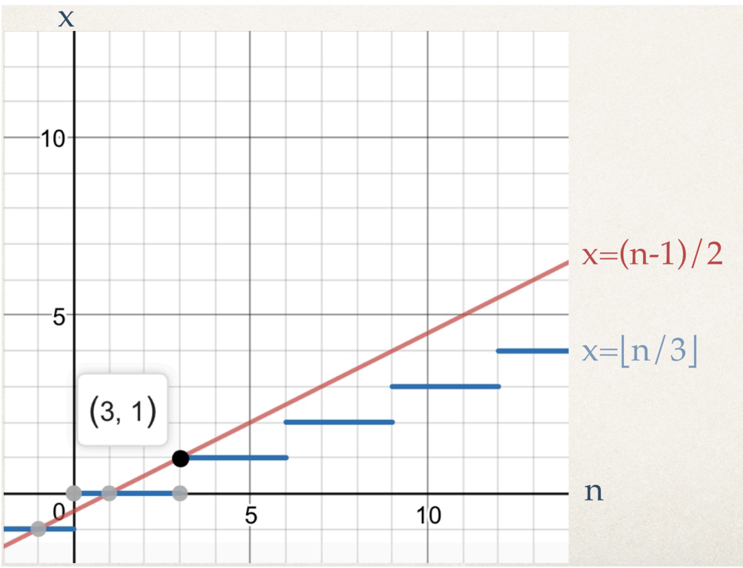

Figure 16.1

Therefore, we can draw a chart to represent the relationship between the number of boxes and least steps needed to find the cat when we can open two boxes per time (Figure 16.2).

Number of Boxes	3	4	5	6	7	8	9	10	11	12	13	14	15	......
Least Steps needed	2	2	2	3	4	4	6	6	6	7	8	8	10	......

Figure 16.2

If the number of boxes is n, least times needed is $2\lfloor N/3 \rfloor - \lfloor mod(N-1,6)/5 \rfloor$.

Similarly, we can also draw the charts when we can open 3 boxes (Figure 16.3) and 4 boxes (Figure 16.4).

Number of Boxes	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	......
Least Steps needed	2	2	2	2	3	4	4	4	4	5	6	6	6	6	7	8	......

Figure 16.3

When the number of boxes is n, least times needed is $2\lfloor (N+1)/5 \rfloor + \lfloor [mod(N+1,5)]/4 \rfloor$.

Number of Boxes	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	......
Least Steps needed	2	2	2	2	2	3	3	4	4	4	4	4	5	5	6	......

Figure 16.4

When the number of boxes is n, least times needed is $2\lfloor (N+2)/7 \rfloor + \lfloor [mod(N+2,7)]/5 \rfloor$.

2.3 Research question 3

How will the arrangement of the boxes (a closed graph or a solid figure) impact the result?

When the boxes are arranged as a closed graph, we can still transfer it into a straight line (Figure 17) but the first box and the last box are connected, which means if the cat is in the first box, it can move to the last box in the next step.

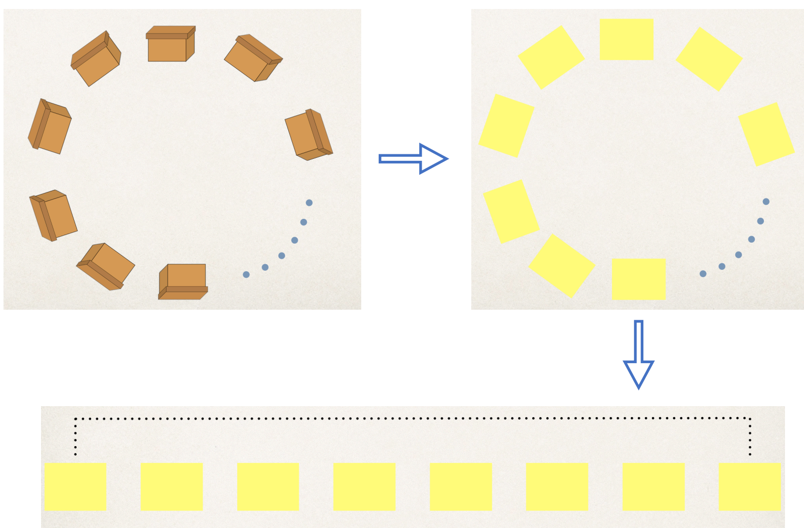

Figure 17

There is no “side” in this case. Therefore, we need to open at least two boxes per time to find the cat. Similarly, it is effective only if we confirm two boxes with another one in between to be empty (Figure 18.1).

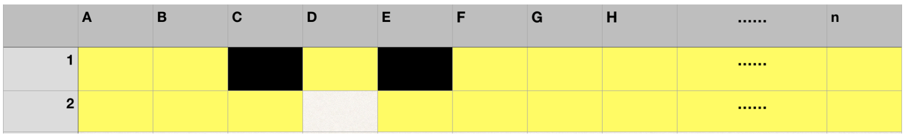

Figure 18.1

Then there are two methods to continue and both will lead to the same result (Figure 18.2).

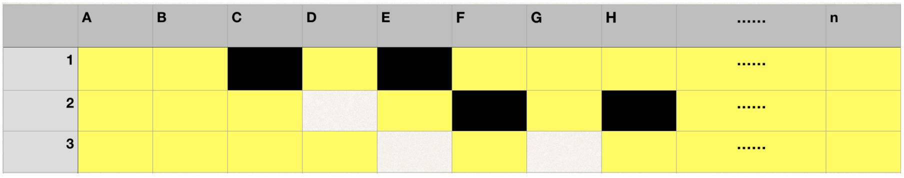

Figure 18.2

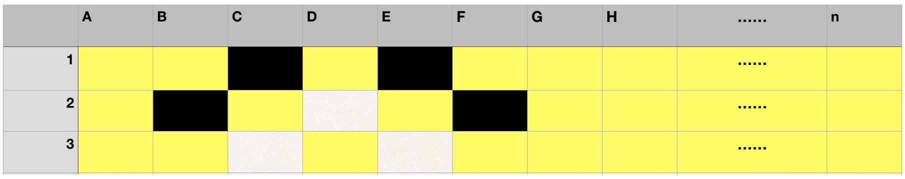

Figure 18.2

We can also draw a chart to represent the relationship between the number of boxes and least steps needed to find the cat when we can open two boxes per time (Figure 18.3).

Number of Boxes	3	4	5	6	7	8	9	10	......
Least Steps needed	2	2	4	4	6	6	8	8	......

Figure 18.3

When the number of boxes is n, least times needed is $2\lfloor (N-1)/2 \rfloor$.

Similarly, we can also draw the charts when we can open 3 boxes (Figure 18.4) and 4 boxes (Figure 18.5).

Number of Boxes	4	5	6	7	8	9	10	11	12	......
Least Steps needed	2	2	2	3	4	4	4	5	6	......

Figure 18.4

When the number of boxes is n, least times needed is $2\lfloor N/4 \rfloor + \lfloor mod(N,4)/3 \rfloor$.

Number of Boxes	5	6	7	8	9	10	11	12	13	14	15	16	17	……
Least Steps needed	2	2	2	2	3	3	4	4	4	4	5	5	6	……

Figure 18.5

When the number of boxes is n, least times needed is $2\lfloor (N+1)/6 \rfloor + \lfloor mod(N+1,6)/4 \rfloor$.

2.4 Conclusion

Research question 1: When the cat is hiding inside one of n boxes in a row, we need at least $2n-4$ steps to find the cat by choosing to open from the second box to the second last box and then from the second last box back to the second box.

Research question 2: It is proven that the general method to find the cat by using the least steps, opening two boxes per time, is to open the boxes from one side to the other: opening the second one and skip the next, opening the fourth one, and in the next step opening the fifth one and seventh one and so on, until all the boxes that the cat might hide inside are separated by an empty box. Observe whether the cat can hide in the second or the second last box. If yes, open from this box and the next box that the cat could hide inside to the other side. If not, just open the boxes from one side to the other.

Number of Boxes	X+1	X+2	……	2X+2	2X+3	2X+4	……	3X-1	3X	3X+1	……
Least Steps needed	2	2	……	2	3	3	……	3	4	4	……

X+1 X-2

We can use a general chart to represent the relationship between the number of boxes and least times needed if we are allowed to open X boxes per time. If the number of boxes is n, least times needed is

$$2\lfloor (N+X-2)/(2X-1) \rfloor + \lfloor mod(N+X-2,2X-1)/(X+1) \rfloor.$$

Research question 3: By transferring the closed graph into a straight line, we can use a similar method. The only difference is that we do not need to consider about the second box or the second last box as this straight line is "cyclic". In the first step, we need to open two boxes, leaving another one in between. Then in the next step, we can either open the boxes from one side of the box which is confirmed to be empty in this step to the other, or open from the empty box to both its two sides.

Number of Boxes	X+1	X+2	......	2X	2X+1	2X+2	......	3X-2	3X-1	3X	......
Least Steps needed	2	2	......	2	3	3	......	3	4	4	......

X X-2

We can use a general chart to represent the relationship between the number of boxes and least times needed if we are allowed to open *X* boxes per time. If the number of boxes is *n*, least times needed is

$$2\lfloor (N+X-3)/(2X-2) \rfloor + \lfloor mod(N+X-3,2X-2)/X \rfloor.$$

3. Others

3.1 New Problems and Directions

In the problems we discussed above, the cat can only move to the adjacent box of the previous one it hided inside. If we change this condition into that the cat can move to any one of *n* consecutive boxes next to the previous box the cat hided inside, the problem could be more complex.

Moreover, we can also arrange the boxes into a solid figure and use several charts to investigate the possible positions of the cat, but to this arrangement, it is quite difficult to find and prove a simplest way.

3.2 Acknowledgements

As this is the first time for us to start a maths project, or even get to know about "project", we faced many difficulties, problems and queries. We need to thank our project mentor Mr. Zong Lixing, as he always answered our questions patiently, no matter about our research content or the structure of our slides. We also improved our analyzing skills a lot with his help.

Moreover, we need to thank the teachers in charge of maths project as they briefed us many times about the requirements of our written report, slides and presentation. They prepared very well and presented clearly every time.

Last but not least, we need to thank the judges in the previous evaluations, as they gave us valuable feedback so that we can know about what we should keep and where we need to improve.

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