A Study on Reducible Polynomial with Integer Coefficients

Yu Shuhuai 3S1 (28) Leader

You Yicheng 3A3 (34)

Gao Jiquan 3S3 (07)

Xu Ruobei 3S2 (29)

Hwa Chong Institution (High School)

Abstract

Factorization of a polynomial consists of writing a polynomial in product form. This study focuses on factorization of quartic polynomial and the irreducibility of general polynomial. We improve the traditional method to factorize quartic polynomials such as the method of undermined coefficients or the remainder theorem by using the knowledge of number theory. Also, we give a necessary condition to determine the irreducibility of a polynomial.

1 Introduction

In algebra, a quartic polynomial is a polynomial of the form:

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e$$

Apparently, $a \neq 0$.

Factorization is to factorize the quartic polynomial to the product of four linear polynomial. Any quartic polynomial is factorizable in complex which is of the form:

$$\mathrm{f(x)} = \mathrm{a}(\mathrm{x} - b)(x - c)(x - d)(x - e) \qquad a, b, c, d, e \in C$$

However, not every polynomial is factorizable in real. There are three cases:

1) Quartic polynomial that is irreducible

   (The irreducibility will be discussed at the last research problem, so the it will not be covered in the first and second research problem)

2) Quartic polynomial with linear factors

For example:

- $\mathrm{f(x)} = \mathrm{a}(x-b)(x-c)(x-d)(x-e) \qquad a, b, c, d, e \ni R$

  Here we give a simple example:

  $$\mathrm{f(x)} = 2(\mathrm{x}-1)\left(x-\frac{2}{3}\right)(x-5)\left(x-\frac{1}{2}\right)$$

- $\mathrm{f(x)} = \mathrm{a}(\mathrm{x}-b)(x-c)(x^2+dx+e) \qquad a, b, c, d, e \ \ni R$

  which (x^2+dx+e) is irreducible in real.

  Here we give a simple example:

  $$\mathrm{f(x)} = 2(\mathrm{x}-1)\left(x-\frac{1}{2}\right)(x^2+x+1)$$

3) Quartic polynomial with two irreducible quadratic factors

$$\mathrm{f(x)} = \mathrm{a}(x^2+bx+c)(x^2+dx+e)$$

which $\Delta_1 = b^2-4c \ < 0$

$$\Delta_1 = d^2-4c \ < 0$$

Here we also give a simple example:

$$\mathrm{f(x)} = 2(x^2+\frac{1}{2}x+4)(x^2+x+\frac{7}{4})$$

Factorizing is a very important skill which helps us to find the roots of the factor of the polynomial. To start, we decided to focus on the **reducible quartic polynomial** first.

In order to find a simplified method that can be used in our math learning to help us with factorizing quartic polynomial, we decide to only research on the **3) case of the reducible quartic polynomial** which no simple method is supplied yet.

Here we give a simple example:

$$x^4 + 4x^3 + 8x^2 + 7x + 4 = (x^2 + x + 1)(x^2 + 3x + 4)$$

Which cannot be further factorized.

1.1 Objective

- To explore the existing methods of factorizing quartic polynomial
- To find out some traits in reducible quartic polynomial.
- To generate some easy ways to factorize reducible quartic polynomial with different number of variables.
-

1.2 Research Problems

- What is the necessary condition for a quartic polynomial to be reducible?
- What are characteristics of the coefficients of those polynomials which can be factorized in that way?
- Generally, what are the necessary condition for a polynomial with integer coefficients to be irreducible?

1.3 Methodology

1.3.1 To read up on resources regarding the factorization of quartic polynomial

from standard books, magazines, papers etc.

1.3.2 To research on related methods of factorizing quartic polynomial.

1.3.3 Find out the advantages and the disadvantages of different methods.

1.3.4 To focus on reducible quartic polynomial and simplified methods. Simplified methods can be used in daily life in order to prove our ability of factorizing.

1.3.5 To prove the necessary condition for whether the reducible quartic polynomial is factorizable

1.3.6 To research on factorizing quartic polynomial with three variables.

1.4 Definitions

- reducible polynomial – In mathematics, a reducible polynomial is, roughly speaking, a non-constant polynomial that can be factored into the product of two non-constant polynomials.

  An example of a reducible quartic polynomial:

$$x^4 + 4x^3 + 8x^2 + 7x + 4 = (x^2 + x + 1)(x^2 + 3x + 4)$$

1.5 Literature Review

1.5.1 The general formula to solve quartic polynomial

There is, in fact, a general formula for solving quartic (4th degree polynomial) equations. As the cubic formula is significantly more complex than the quadratic formula, the quartic formula is significantly more complex than the cubic formula. Wikipedia's article on quartic functions has a lengthy process by which to get the solutions, but does not give an explicit formula.

Let:

$$p_1 = 2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace$$

$$p_2 = p_1 + \sqrt{-4(c^2 - 3bd + 12ae)^3 + p_1^2}$$

$$p_3 = \frac{c^2 - 3bd + 12ae}{3a\sqrt[3]{\frac{p_2}{2}}} + \frac{\sqrt[3]{\frac{p_2}{2}}}{3a}$$

$$p_4 = \sqrt{\frac{b^2}{4a^2} - \frac{2c}{3a} + p_3}$$

$$p_5 = \frac{b^2}{2a^2} - \frac{4c}{3a} - p_3$$

$$p_6 = \frac{-\frac{b^2}{a^2} + \frac{4bc}{a^2} - \frac{8d}{a}}{4p_4}$$

Then:

$$x_1 = -\frac{b}{4a} - \frac{p_4}{2} - \frac{\sqrt{p_5 - p_6}}{2}$$

$$x_2 = -\frac{b}{4a} - \frac{p_4}{2} + \frac{\sqrt{p_5 - p_6}}{2}$$

$$x_3 = -\frac{b}{4a} + \frac{p_4}{2} - \frac{\sqrt{p_5 - p_6}}{2}$$

$$x_4 = -\frac{b}{4a} + \frac{p_4}{2} + \frac{\sqrt{p_5 - p_6}}{2}$$

This method can factorize all quartic polynomial but we think this way is too complicated. It is impossible for us to memorize and use.

Address: https://math.stackexchange.com/questions/785/is-there-a-general-formula-for-solving-4th-degree-equations-quartic

1.5.2 The method of undetermined coefficients

From the above literature review, we get that any

$$\mathrm{a}X^4 + bX^3 + cX^3 + dX + \mathrm{e} = 0 \quad (\mathrm{ae} \neq 0)$$

can be transferred into

$$y^4 + Ay^2 + By + C = 0$$

Case 1 (B≠0):

$$y^4 + Ay^2 + By + C = (y^2 + p \cdot y + m)(y^2 - p \cdot y + n)$$

(since the coefficient of Y^3 is 0, so the sums of the coefficients of Y^1 in the two quadratic trinomials should be 0)

The P, M, N are undetermined coefficients of the equation

The following step is to solve the simultaneous equations:

$$\begin{cases} \mathrm{A} = \mathrm{m} + \mathrm{n} - p^2 \\ B = (n - m) \times p \\ c = m.n \end{cases}$$

Then finally solve two equations

$$y^2 + py + m = 0$$

$$y^2 - py + n = 0$$

Get the four roots.

Case 2 (B=0):

The equation is $\quad y^4 + Ay^2 + C = 0$

$$y^2 = \frac{-A \pm \sqrt{A^2 - 4C}}{2}$$

$$\mathrm{y} = \pm\sqrt{\frac{-A \pm \sqrt{A^2 - 4C}}{2}}$$

This method seems to be a little better than the first one, the train of thinking is a lot clearer. But actually in some cases, we still need to solve a cubic equation. The method is undetermined coefficients, using the concepts of equations, turns the quartic equation into two quadratic equations which are much easier to solve.

Address: http://www.sosmath.com/algebra/factor/fac12/fac12.html

1.5.3 Remainder theorem

In algebra, the polynomial remainder theorem or little Bézout's theorem is an application of Euclidean division of polynomials. It states that the remainder of the division of a polynomial f(x) by a linear polynomial (x-a) is equal to f(a). In particular, (x-a) is a divisor of f(x) if and only if f(a)=0.

For example, let

$$\mathrm{f(x)} = 5x^3 + 4x^2 - 12x + 1$$

the remainder of $\frac{f(x)}{x-3}$ is $5 \times 3^3 + 4 \times 3^2 - 12 \times 3 + 1 = 136.$

Proof: Set (x-a) divided by a polynomial f(x), the quotient $\mathrm{Q(x)}$ and the remainder R, according to the above property, can list the following identities: $f(x) = (x - a)Q(x) + R$. Hence $f(a) = (a - a)Q(x) + R = R$, so it can be proved.

The advantages of this method are that we can find the remainder of a complicated polynomial with small degree faster and more easily.

The disadvantages of this method are that if the degree is big, it may be very troublesome to calculate expressions like 5^10.

2 Results

2.1 What is the necessary condition for a quartic polynomial to be reducible?

For this problem, we did the research on the necessary condition for a quartic polynomial to be reducible. Quartic polynomials refer to polynomials with highest degree 4. The general formula of a quartic polynomial is

If a quartic polynomial has two irreducible quadratic factors, then it can also be written in the form of

$$(x^2 + nx + m)(x^2 + px + q)$$

After expanding it, we can get

$$x^4 + px^3 + nx^3 + npx^2 + mx^2 + qx^2 + nqx + mpx + mq$$

$$= x^4 + (n + p)x^3 + (q + m + np)x^2 + (nq + mp)x + mq$$

Compared to the general form of quartic polynomial, we can know that

$$\begin{cases} a = n + p \\ b = q + m + np \\ c = nq + mp \\ d = mq \end{cases}$$

The main idea of the following process is to cancel some of the variables. We want to get a simple expression which can help us to determine the irreducibility of the polynomial.

So, we defined

$$T = a^2 - 4b + 4(m + q)$$

the necessary condition for the quartic polynomialis that we could find particular value s of m, q which makes T a perfect square.

To prove it, as $a = n + p$, $b = q + m + np$

$$\mathrm{T} = (n + p)^2 - 4(q + m + np) + 4(m + q) = (n - p)^2$$

Since n and p are all integers, T must be a perfect square. Therefore, we found that the necessary condition for $f(x)$ to be reducible is that T is a perfect square. As

$$T = (n - p)^2$$

$$\sqrt{T} = |n - p|$$

$$a = n + p$$

We could know that $(n, p) = \left(\frac{a+\sqrt{T}}{2}, \frac{a-\sqrt{T}}{2}\right)$.

In this way, we were able to compute the value of all $\mathrm{n}, \mathrm{p}, \mathrm{m}, \mathrm{q}$. However, we still need to check the answer. Since $c = nq + mp$, if the value of $nq + mp$ is the value of coefficient c in the question, then we successfully factorize the polynomial.

Here we give an example.

E.g. Factorize

$$x^4 + 11x^3 + 48x^2 + 125x + 75$$

As $\mathrm{mq} = 75$ and $\mathrm{m, q}$ are integers, $(m, q) = (1,75)\ or\ (3,25)\ or\ (5,15)$. Then we will discuss this case by case.

Case 1:

When $(m, q) = (1,75)$

$T = a^2 - 4b + 4(m + q)$

$= 11 \times 11 - 4 \times 48 + 4 \times (1 + 75)$

$= 233$

As 233 is not a perfect square, this case cannot work.

Case 2:

When $(m, q) = (3,25)$

$T = a^2 - 4b + 4(m + q)$

$= 11 \times 11 - 4 \times 48 + 4 \times (3 + 25)$

$= 41$

As 41 is not a perfect square, this case also cannot work

Case 3:

When $(m, q) = (5,15)$

$T = a^2 - 4b + 4(m + q)$

$= 11 \times 11 - 4 \times 48 + 4 \times (5 + 15)$

$= 9$

$= 3^2$

As 9 is a perfect square, we would further discuss this case.

Hence $(m,q) = (5,15)$. As we mentioned, $(n,p) = \left(\frac{a+\sqrt{T}}{2}, \frac{a-\sqrt{T}}{2}\right)$. We can calculate that

$$(n,p) = \left(\frac{11+3}{2}, \frac{11-3}{2}\right) = (7,4)$$

However, n and p can swap with each other, so $(\mathrm{n},\mathrm{p}) = (7,4)$ or $(4,7)$. We check the value of $c = nq + mp$. When $\mathrm{n} = 4, \mathrm{p} = 7$, $nq + mp = 5 \times 7 + 15 \times 4 = 95 \neq c$, thus this case cannot work. When $\mathrm{n} = 7, \mathrm{p} = 4$, $nq + mp = 5 \times 4 + 15 \times 7 = 125 = c$. Hence only $(\mathrm{n},\mathrm{p}) = (7,4)$ satisfy the condition.

Therefore, the answer to this question is

$$\mathrm{x}^4 + 11\mathrm{x}^3 + 48\mathrm{x}^2 + 125\mathrm{x} + 75 = (\mathrm{x}^2 + 7\mathrm{x} + 5)(\mathrm{x}^2 + 4\mathrm{x} + 15)$$

2.2 What are characteristics of the coefficients of those polynomials which can be factorized in that way?

In the problem 1 we discussed about the necessary condition for quartic polynomial to be reducible and give a possible method to factorize it. In problem 2, we will further simplify the factorization process by using number theory.

As we defined

$$T = a^2 - 4b + 4(m+q)$$

the necessary condition for the quartic polynomialis that we can find particular values of m, q which makes T a perfect square. Since T is a perfect square, we would like to discuss the property of a perfect square. We know that

$$a^2 \equiv 1 \; or \; 0(mod\ 4)$$

It is easy to prove. If a is an even number, it is clear that a^2 is the multiple of 4. If a is an odd number, let a be $2k+1$, we had

$$a^2 = (2k+1)^2 = 4k^2 + 4k + 1 \equiv 1(mod\ 4)$$

We found that in the expression of T, the coefficients of b and $(m+q)$ are 4 and -4. If a is an even number, then we can have common factor 4. We assume a is an even number. Let $a = 2k$, then we have

$$T = 4k^2 - 4b + 4(m+q)$$

$$\frac{T}{4} = k^2 - b + m + q$$

It is clearly that $\frac{T}{4}$ is also a perfect square. Since $d = mq$, we discuss the property of d first. There are four cases, d is a prime number, or $d \equiv 1,2,3(\mathrm{mod}\ 4)$ respectively. Since $k^2 - b + m + q \equiv 1 \text{ or } 0(\mathrm{mod}\ 4)$, by discussing the modulo of d, k^2, we could find that for certain values of b, d, $k^2 - b + m + q$ cannot be a perfect square. We showed our results by the following form.

	Condition of d	**Condition of m and q**	**Condition of k^2**	**Conclusion**
Case one	prime	$m = 1, q = d$	$k^2 \equiv 1(\mathrm{mod}\ 4)$	$(d-b) \equiv 0 \; or \; 1(mod\ 4)$
			$k^2 \equiv 0(\mathrm{mod}\ 4)$	$(d-b) \equiv 0 \; or \; 1(mod\ 4)$
Case two	$d \equiv 1\ (mod\ 4)$	$m \equiv q \equiv 1\ (mod\ 4)$	$k^2 \equiv 1(\mathrm{mod}\ 4)$	$b \equiv 0 \; or \; 3(mod\ 4)$
			$k^2 \equiv 0(\mathrm{mod}\ 4)$	$b \equiv 0 \; or \; 1(mod\ 4)$
		$m \equiv q \equiv 3\ (mod\ 4)$	$k^2 \equiv 1(\mathrm{mod}\ 4)$	$b \equiv 0 \; or \; 3(mod\ 4)$
			$k^2 \equiv 0(\mathrm{mod}\ 4$	$b \equiv 0 \; or \; 1(mod\ 4)$

Case three	$d \equiv 2\ (mod\ 4)$	$m \equiv 1(mod\ 4)$	$k^2 \equiv 1(\text{mod } 4)$	$b \equiv 2\ or\ 3(mod\ 4)$
		$q \equiv 2(mod\ 4)$	$k^2 \equiv 0(\text{mod } 4)$	$b \equiv 0\ or\ 3(mod\ 4)$
		$m \equiv 2(mod\ 4)$	$k^2 \equiv 1(\text{mod } 4)$	$b \equiv 1\ or\ 0(mod\ 4)$
		$q \equiv 3(mod\ 4)$	$k^2 \equiv 0(\text{mod } 4)$	$b \equiv 1\ or\ 2(mod\ 4)$
Case four	$d \equiv 3\ (mod\ 4)$	$m \equiv 1\ (mod\ 4)$	$k^2 \equiv 1(\text{mod } 4)$	$b \equiv 0\ or\ 3(mod\ 4)$
		$q \equiv 3(mod\ 4)$	$k^2 \equiv 0(\text{mod } 4)$	$b \equiv 0\ or\ 1(mod\ 4)$

Polynomial cannot be factorized in those conditions. Here we give an example to illustrate our idea.

E.g.

Factorize $x^4 + 12x^3 + 52x^2 + 125x + 61$

We could see that $\text{a} = 12$ is an even number, so we could apply our ideas. We also found that $\text{d} = 61 \equiv 1(\text{mod } 4), \text{b} = 52 \equiv 0(\text{mod } 4)$, by looking through the form, we know the polynomial cannot be factorized in this condition. Hence, $x^4 + 12x^3 + 52x^2 + 125x + 61$ is irreducible.

2.3 Generally, what are the necessary condition for a polynomial with integer coefficients to be reducible?

In the third research problem, we want to further discover the general necessary condition for a polynomial with integer coefficients to be irreducible.

In algebra, a polynomial of degree n can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

We claim that for infinite value of $f(x)$, if we can find $2n - 1$ integers which make the value of $f(x)$ prime numbers, then $f(x)$ is irreducible. Now we prove the claim by contradiction.

Let those integers be

$$a_1, a_2, a_3 \cdots a_{2n-2}, a_{2n-1}$$

and

$$f(a_1) = p_1, f(a_2) = p_2 \cdots, f(a_{2n-2}) = p_{2n-2}, f(a_{2n-1}) = p_{2n-1}$$

where

$$p_1, p_2, \cdots, p_{2n-2}, p_{2n-1}$$

are all prime numbers. Also, we let $g(x)$ to be a factor of $f(x)$ with highest degree, it is clear that the degree of $g(x)$ is equal to or less than $n - 1$,

We notice that if $g(x)|f(x)$ for any integer value, thus

$$g(x_i)|p_i \quad i \in \{1 \leq x \leq 2n - 1 | \, x \in Z\}$$

When $g(x)$ divides a primes number, it means $|g(x)|$ is either 1 or p itself. However, in this case, $|g(x)| < |f(x)|$. It is easy to see that $f(x) = g(x) \cdot h(x)$, $h(x)$ can be any polynomial with smaller degree than $g(x)$ in real. Because the coefficients of $h(x)$ are all integers, so $|h(x)| > 0 \; or \; |h(x)| = 1$. This imply that $|g(x)| \leq |f(x)|$. Therefore, we conclude that $g(x)$ can only be $1 \text{ or } -1$.

By Pigeon-hole Principle, for $2n-1$ g(x) values which are either 1 or -1, we are able to find n g(x) with the same values. Without the loss of generality, we assume $g(x_i) = 1$ for $i \in \{1,2,3 \cdots n-1, n\}$. It is clear that $x_1, x_2, \cdots, x_{n-1}, x_n$ are n different roots for $g(x) = 1$. However, as we mentioned before, g(x) is a at most $n-1$ degree polynomial, which means it could at most have $n-1$ roots. This contradict to the n different roots g(x) has, which shows $f(x)$ is irreducible.

Here we give a simple example to illustrate our idea.

E.g. To show x^2+3x+1 is irreducible in real.

The polynomial is a degree 2 polynomial; thus, we only need to find $2 \times 2 - 1 = 3$ integers that makes the value of the polynomial a prime number.

Let $f(x) = x^2+3x+1$. We find $f(1) = 5, f(2) = 11, f(3) = 19$

Since 5, 11, 19 are all prime numbers, x^2+3x+1 is irreducible in real.

$$a_1, a_2, a_3 \cdots a_{2n-2}, a_{2n-1}$$

$$f(a_1) = p_1, f(a_2) = p_2 \cdots, f(a_{2n-2}) = p_{2n-2}, f(a_{2n-1}) = p_{2n-1}$$

$$p_1, p_2, \cdots, p_{2n-2}, p_{2n-1}$$

3 Extension

Factorization of polynomial with high degree is still a problem needed to be solved and it is mainly computed by computer. Also, factorization is useful in the compute of mathematic modelling, such as weather system. We hope that our results could be further extended to improve the efficiency of existed method and improve our skills of factorization.

4 Acknowledgement

We thank our mentor Mr. Zong for his continuous support and guidance.

5 References

5.1 StackExchange (2010). *Is there a general formula for solving 4th degree equations (quartic)?* [online] Last edited 30 July 2012 at: https://math.stackexchange.com/questions/785/is-there-a-general-formula-for-solving-4th-degree-equations-quartic

5.2 S.O.SMath. *The "Quartic Formula"* [online] Last edited 30 January 1997 at: http://www.sosmath.com/algebra/factor/fac12/fac12.html

5.3 Shuhong Gao, "Absolute Irreducibility of Polynomials Via Newton Polytopes" Department of Mathematical Science, Clemson University at https://pdfs.semanticscholar.org/6d80/fcdc480dcd6554950de1679fe603036c8c54.pdf?tdsourcetag=s_pctim_aiomsg

5.4 R. Thangadurai,"Irreducibility of Polynomials Whose Coefficients are Integers", Harish-Chandra Research Institute at: http://www.hri.res.in/~thanga/papers/mnl.pdf?tdsourcetag=s_pctim_aiomsg

5.5 黄嘉盛. 不可约多项式的判定及应用[M]. 高等代数

https://wenku.baidu.com/view/8590771ffbd6195f312b3169a45177232f60e438.html

5.6 权小刚. 利用分解因式法解一元四次方程[J]. 考试周刊 2021.20

5.7 戴中林. 一元四次整系数多项式的因式分解法[C]. 大学数学 2016.12

https://wenku.baidu.com/view/1bb565859a89680203d8ce2f0066f5335a8167df.html