Math in a paper plane

Group Number 8-17

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1.Introduction

Paper plane is a classical toy aircraft with a long history. In 1930, Jack Northrop, the co-founder of Lockheed Corporation used paper planes as test models for larger aircrafts. It is quite easy to fly a paper plane, the player just need to throw it out by hand.

Figure 1a A classical type of paper plane

While some people can fold and throw a paper plane that flies far, some people are unable to do so. Our research aims to figure out the the effect of how people fold and throw paper planes on the flying distance of the paper plane, and hence find the optimum strategy to fold and throw a paper plane to make it fly the farthest.

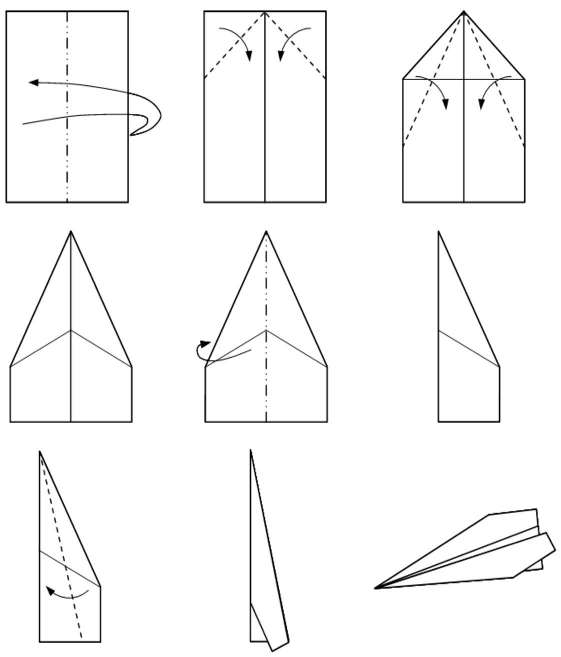


Figure 1b A typical paper plane

1.1 Objective

- Find how to fold a paper plane to fly it the farthest.
- Find how to throw a paper plane to fly it the farthest.

1.2 Research Questions

- How to build an ideal model of the paper plane?
- How to add air resistance into the model ?
- How to add angle of takeoff into the model ?

2 Methodology

Force analysis is frequently used in our research and is a basic method of relating an object's motion to its force acted upon. Using a free body diagram, all forces affecting motion of the paper plane are considered as a whole. In this process, Newton's second and third law are also used to quantify the motion.

The free body diagram of a paper plane is shown below as Figure 2a.

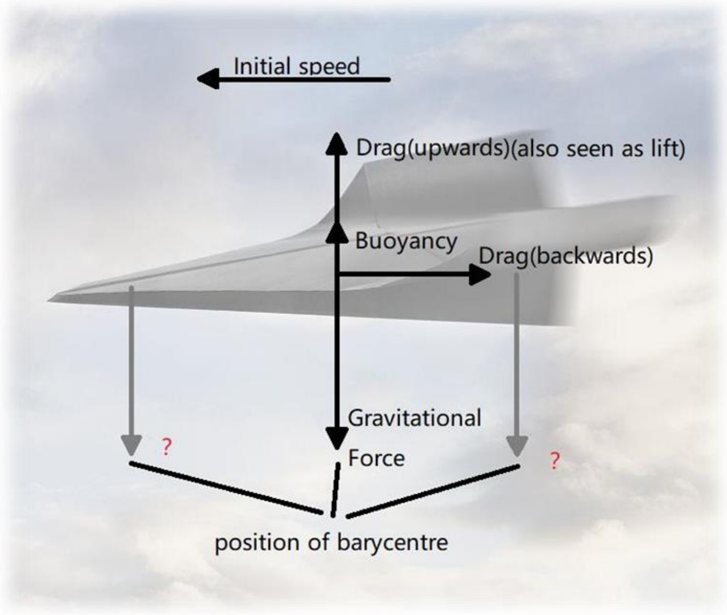


Figure 2a

We consider the motion of the plane in two dimensions, which are the X-axis and Y-axis, it means that we do not consider the plane flying to the left or right due to aerodynamic matters.

On the X axis, it has a initial velocity pointing at front when the plane leave one's hand, hence it faces a drag resulted by air resistance backwards (Figure 2b).

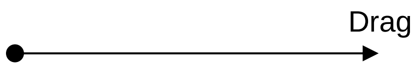


Figure 2b

On the Y axis, the plane falls due to gravitational pull, hence it faces a drag upwards due to air resistance, as the plane take up the space of some air, it faces a slight air buoyancy.

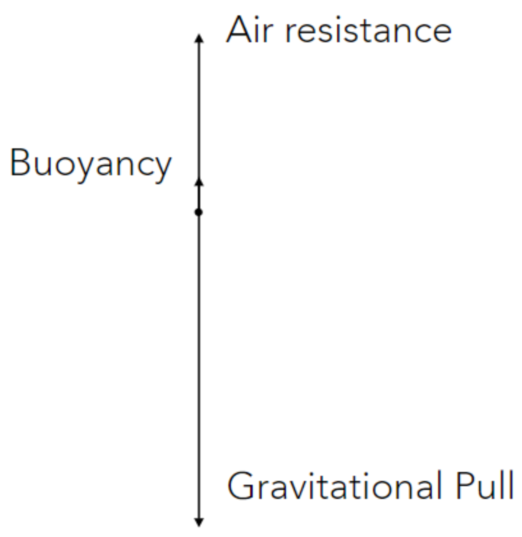


Figure 2c Force analysis on the y-axis

Because the position of the barycentre is not at the centre of the plane, the plane may flip slightly and face a bigger air resistance and reduce its speed. (Figure 2c)

We build three models to represent the gliding process of the paper plane, corresponding to our three research questions.

In our ideal model which is also the first model of a paper plane, we do not consider air resistance, air buoyancy, and position of barycenter, the paper plane is only acted upon gravity. In this model, the paper plane can be seen as a mass point, the model is the start of our analysis.

Air resistance plays a very significant role in the paper plane's gliding, as we have mentioned before in the force analysis before, horizontal air resistance will reduce the velocity at front; vertical resistance will reduce the velocity when the plane climbs when thrown (we consider this situation in our third research problem), and when the plane is falling down.

Hence we add air resistance into our second model of paper plane. Because the density of paper is a lot higher than that of air, we do not consider the air buoyancy in our second model also. Compare to the first model, the paper plane will reduce its horizontal velocity very slightly because its very small horizontal projection area (we will give more detailed explanation of this in our second research results), but its vertical velocity will significantly reduce, (by half, according to our calculation which will be presented later in the second research results as well), the plane will stay in the air for a significantly longer time, hence paper plane in the second model will fly significantly farther than that in the first model.

Angle of takeoff is also of significance in the process, normally we throw out the paper plane diagonally, which brings it to a higher altitude to fall. As stated before in our force analysis, position if barycenter may cause the whole plane to flip, presenting a significantly larger projection area in the air, hence slowing down the plane. An angle of takeoff will bring the paper plane to a linger range, and a inappropriate position of barycenter will reduce the range. We take these two factors into consideration in out third model of paper plane motion. If the paper plane has a proper position of barycenter, the paper plane in our third model will fly significantly farther than that in the second model.

This is the precise way of doing force analysis of a paper plane.

There is a huge difference between the paper plane we fold and the actual ones. The difference between a paper plane and a actual plane lies in the wings of the actual plane are designed, so as to make the air which flows over the wings move faster than the air flows underneath, as shown in Figure 2d.

The design creates a lower pressure on top compared to bottom of the wings. The pressure difference allows planes to have an lift force to prevent them from falling to the ground.

We do not consider the air buoyancy of our paper planes because the thickness of the paper plane wing is negligible, hence the difference in speed of air flowing along the upper and lower sides of the wings is negligible, the pressure difference to lift the plane is negligible.

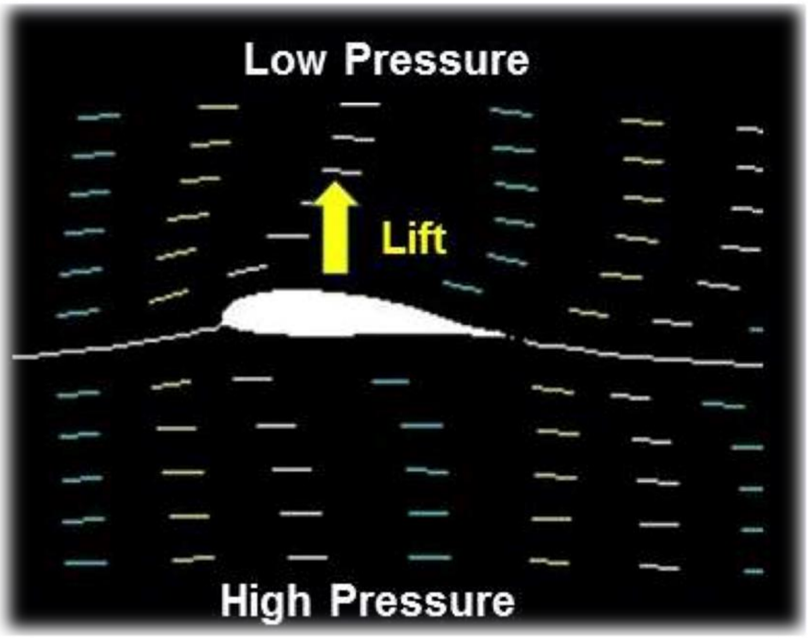


Figure 2d An actual wing's design

3. Literature Review

A Taiwanese study group has done researches on this topic before. They conducted Physics experiments to investigate the effect of paper types, angle of takeoff, angle of wings and position of barycentre on the flying distance of a paper plane.

Figure 3a Launching device used by the research group

According to their results, as shown in Figure 3b, flying distance decreases when angle of takeoff increases from 0 degree to 30 degrees, when two clips are added at the front, the plane flies the farthest.

	Position of barycentre - flying distance/cm										
	1	2	3	4	5	6	7	8	9	10	Average
No clip added	842	694	708	944	925	860	820	889	930	822	843.4
Two clips at the front	883	842	923	853	883	875	991	973	862	985	907
Two clips at the back	220	198	212	153	171	165	186	188	145	196	183.4

Angle of elevation	Distance/cm										
	1	2	3	4	5	6	7	8	9	10	Average
0 degree	460	521	514	555	500	494	544	480	441	486	499.5
15 degrees	560	478	303	422	373	489	309	451	400	230	401.5
30 degrees	231	252	330	257	285	250	321	368	250	257	280.1

Figure 3b Part of results of the research group's work

However, the research is based on experiments and hence lacks precision which is especially belong to mathemetics.

Also we find a family experiment on the website http://blog.sina.com.cn/jiaxin2016 that research on the effect of paper density on the flying distance of the plane. The results showed that 70 grams per square meter of paper is the optimum density, which is also the density of normal printing paper.

Figure 3c the color of paper representing different density

paper density (grams/square meter)	50	70	80	90
range 1 /cm	320	665	470	400
range 2 /cm	383	588	480	420
average /cm	352	627	475	410

Figure 3d results of the family experiment

We will use mathematical ways to find out the optimum solution to fold and throw a paper plane.

4. Known conditions

In this research, we fold a paper plane using an A4 paper. An A4 paper is 210mm wide and 297mm long, and it has a density of 70gram/ square meter. Known facts of an A4 paper are listed below.

Area of an A4-size paper	$S = 0.2100\mathrm{m} \times 0.2970\mathrm{m} = 0.06237\mathrm{m}^2$
Mass of the paper plane	$m=70.00g/m^2\times0.6237\mathrm{cm}^2=4.366g$
Gravity of the paper plane	$G=\mathrm{mg}=4.366g\times9.781m/s^2=0.04270N$
Height when takeoff	2.000m

Figure 4

5.Results

5.1 The first research question: How to build an ideal model of the paper plane?

In this model we consider gravity of the plane, which is 0.0427N as stated in chapter 4.We consider gravity of the plane as the only force acting on the paper plane, as shown in Figure 5a and calculate time taken for the paper plane to fall on the ground.

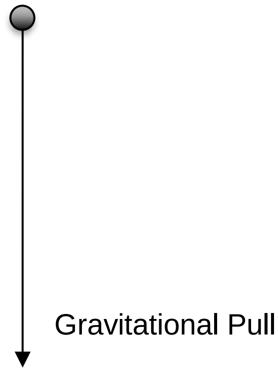


Figure 5a The free body diagram of the paper plane in the first model

Vertically, according to the basic equation, the falling distance h, equals to initial velocity, v0, times time, t; plus acceleration, a, multiplied by the square of time divided by 2.

$$h = v_0 t + \frac{at^2}{2}$$

we derive

$$h = \frac{a \times t^2}{2}$$

Using height h and acceleration a to describe t, we get

$$t = \sqrt{\frac{2h}{a}}$$

Because acceleration here equals to the acceleration due to gravity, we finally get the expression between time taken for the plane to fall on the ground and height as well as velocity when it leaves one's hand.

$$D = v \times \sqrt{\frac{2h}{g}}$$

If it the constant initial speed is 2 meter per second, the paper plane will travel approximately 1.3 meters. The result may contradict with the common knowledge, this is because we normally throw a paper plane out diagonally so that the paper plane falls from a greater height.

5.2 The second question: How to add the effect of air into the model?

We add air buoyancy and air resistance into consideration. In this case, the position of center of gravity is ignored (Figure 5b).

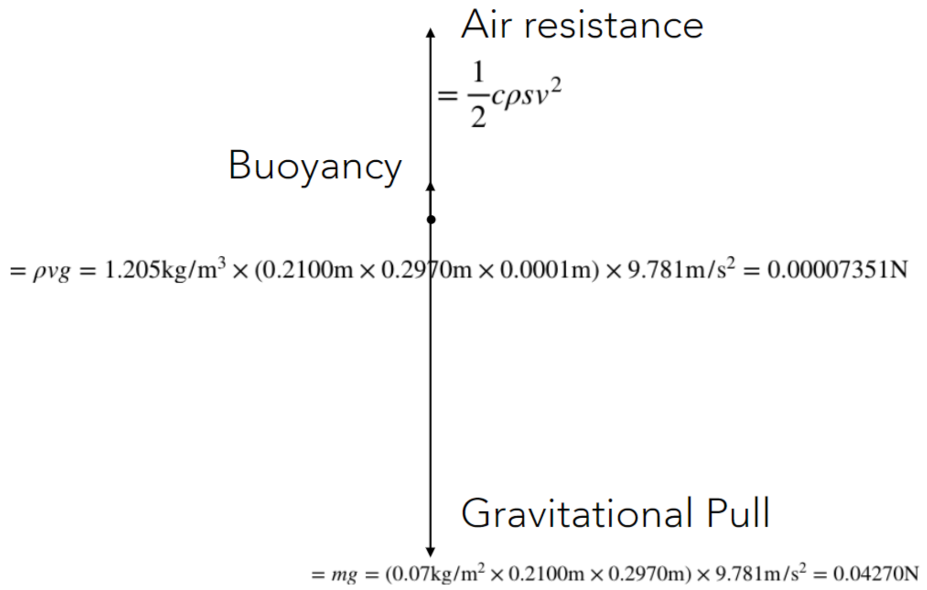


Figure 5b Free body diagram

Air buoyancy is equal to the space taken up by the paper plane in the air, V multiplied by air density, ρ and gravitational acceleration, g. It is around 0.00007N, comparing to the gravitational pull of 0.0427N, air buoyancy is negligible.

$$\text{Buoyancy} = \rho v g = 1.205\text{kg/m}^3 \times (0.2100\text{m} \times 0.2970\text{m} \times 0.0001\text{m}) \times 9.781\text{m/s}^2 = 0.00007351\text{N}$$

The air resistance equals to drag efficiency C, multiplied by air density ρ, the vertical projection area of the paper plane, and the square of vertical speed, divided by 2.

$$\text{Air resistance} = \frac{1}{2} c\rho s v^2$$

The first letter c, is a constant, drag coefficient.

Different shapes has different drag efficiency, as shown in Figure 5c, we found that the shape of vertical projection between angled cube and a cube, we suppose it to be 1.000.

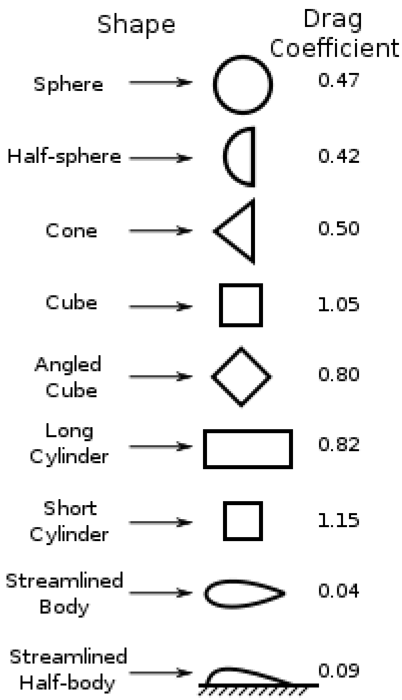


Figure 5c Drag coefficients of different shapes

The second constant is density of air, ρ, which is 1.205 kilogram per cube meter at room temperature.

There are two variables in the equation. One is S, which means vertical area of projection. Different ways of folding decide different values of S. It changes between 0 and 0.0624 square meters, which is the area of one A4 paper.

$$0\mathrm{m}^2 < \boldsymbol{S} \leq 0.0624\mathrm{m}^2$$

The other variable is V, vertical speed. It is defined as v equals to u, which is the vertical initial velocity which is actually 0, plus a*t. Here comes another variable, a

A is the vertical acceleration. According to Newton 2nd law, a equals to the resultant force on that direction over the mass of the object - It equals to weight of the paper plane minus the vertical drag minus buoyancy over mass of paper plane.

$$F = ma$$

$$a = \frac{F_R}{m} = \frac{G - \text{Drag} - \text{Buoyancy}}{m}$$

We can see that the variable drag, is again led into the equation

Because the whole process of motion is related to complicated calculus, we have created a computer programme to simulate calculus and scan 1000 times per second to get the paper plane’s acceleration, speed and height at every moment. The source code and definitions of variables are listed in Annex 1. The results are listed in Figure 5d. In the table s represents the vertical projection area of the paper plane, and t represents the range.

0.031	1.43296
0.032	1.45247
0.033	1.47078
0.034	1.49042
0.035	1.51064
0.036	1.52791
0.037	1.54606
0.038	1.56558
0.039	1.58202
0.04	1.59982
0.041	1.61876
0.042	1.63418
0.043	1.65176
0.044	1.6703
0.045	1.68722
0.046	1.70286
0.047	1.71826
0.048	1.73436
0.049	1.75007
0.05	1.76803
0.051	1.782
0.052	1.79826
0.053	1.8141
0.054	1.83076
0.055	1.84772
0.056	1.86209
0.057	1.87906
0.058	1.89059
0.059	1.90491
0.06	1.92239
0.061	1.93762

s	t
0.001	0.670402
0.002	0.700823
0.003	0.731575
0.004	0.762437
0.005	0.793198
0.006	0.82372
0.007	0.853881
0.008	0.883581
0.009	0.912771
0.01	0.941409
0.011	0.969497
0.012	0.997005
0.013	1.02398
0.014	1.05041
0.015	1.07627
0.016	1.10173
0.017	1.12652
0.018	1.15096
0.019	1.1749
0.02	1.19861
0.021	1.22107
0.022	1.24371
0.023	1.26643
0.024	1.28807
0.025	1.30999
0.026	1.33106
0.027	1.35241
0.028	1.37262
0.029	1.39283
0.03	1.4123

Figure 5d

A graph of time in the air t, against vertical projection area, s, is plotted, as shown in Figure 5e. From the graph, t increases while S increases

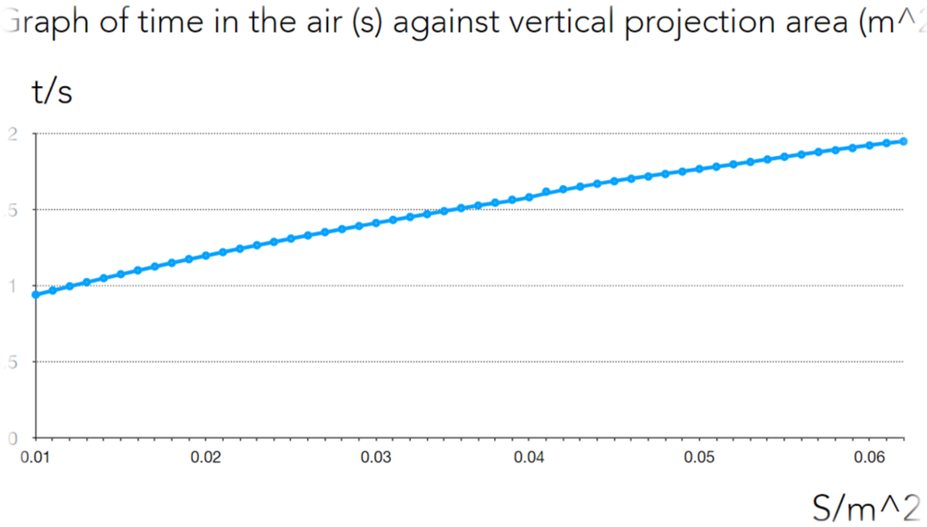


Figure 5e

To further figure out the relationship between time in the air and area of vertical projection, we substitute all the variables and constants into the equation we have.

$$a = \frac{F_R}{m} = \frac{G - \text{Drag} - \text{Buoyancy}}{m}$$

$$= \frac{0.0427\text{N} - \frac{1}{2}c\rho s v^2 - 0.00007351\text{N}}{m}$$

$$= \frac{0.0427\text{N} - \frac{1}{2} \times 1.000 \times 1.205\text{kg/m}^3 \times s(u + at)^2 - 0.00007351\text{N}}{(4.366 \div 1000)\text{kg}}$$

$$\frac{4}{t^2} = \frac{0.0427 - 0.6025s(\frac{4}{t})^2 - 0.00007351}{0.004366}$$

We input the equation into desmos (t=y,s=x), as shown in Figure 5f

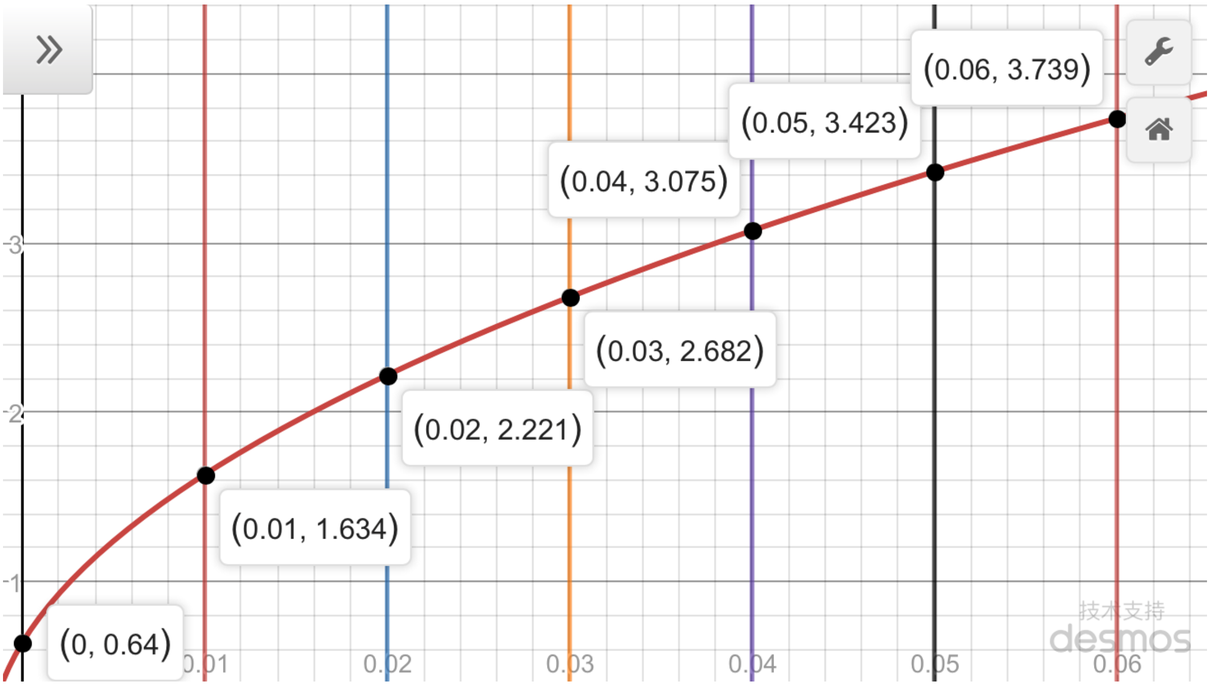


Figure 5f Part of the graph obtained in the first quadrant.

This trend is more close to the real situation in our lives. From the equation and the graph we can finally get that time paper plane in the air is only related with its area of vertical projection. Time in the air increases when area of vertical projection increases, the plane flies farther.

5.3 The third question: How to add angle of takeoff into the model.

The paper plane leaves the hand diagonally, hence has a higher distance to fall on the ground, it will fly even farther.

In our third research question, when a paper plane leaves one's hand diagonally, because it has an oblique initial velocity, it is faced with air resistance downwards due to its velocity upwards and air resistance backwards. The process of force analysis is shown in Figure 5g.

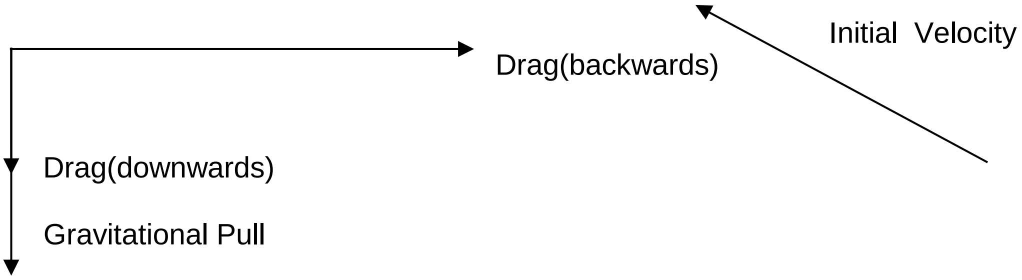


Figure 5g

In this research question, we assume that we are using a paper plane with a constant total area of wing

$$s = 0.04\mathrm{m}^2$$

We have built a function to simulate its movement. The procedure is listed below.

$$a = \frac{F_R}{m} = \frac{G - \mathrm{Drag} - \mathrm{Buoyancy}}{m}$$

$$= \frac{0.0427\mathrm{N} - \frac{1}{2}c\rho s v^2 - 0.00007351\mathrm{N}}{m}$$

$$= \frac{0.0427 - \frac{1}{2} \times 1.000 \times 1.205 \times 0.04\cos\theta(-2\sin\theta + \frac{2\sin\theta t + 2}{0.5t^2} \times t)^2 - 0.00007351}{4.366 \div 1000}$$

$$\frac{2\sin\theta t + 2}{0.5t^2} = \frac{0.0427 - \frac{1}{2} \times 1.000 \times 1.205 \times 0.04\cos\theta(-2\sin\theta + \frac{2\sin\theta t + 2}{0.5t^2} \times t)^2 - 0.00007351}{4.366 \div 1000}$$

We input this equation to desmos (t=y, θ=x) and got the graph (Figure 5h)

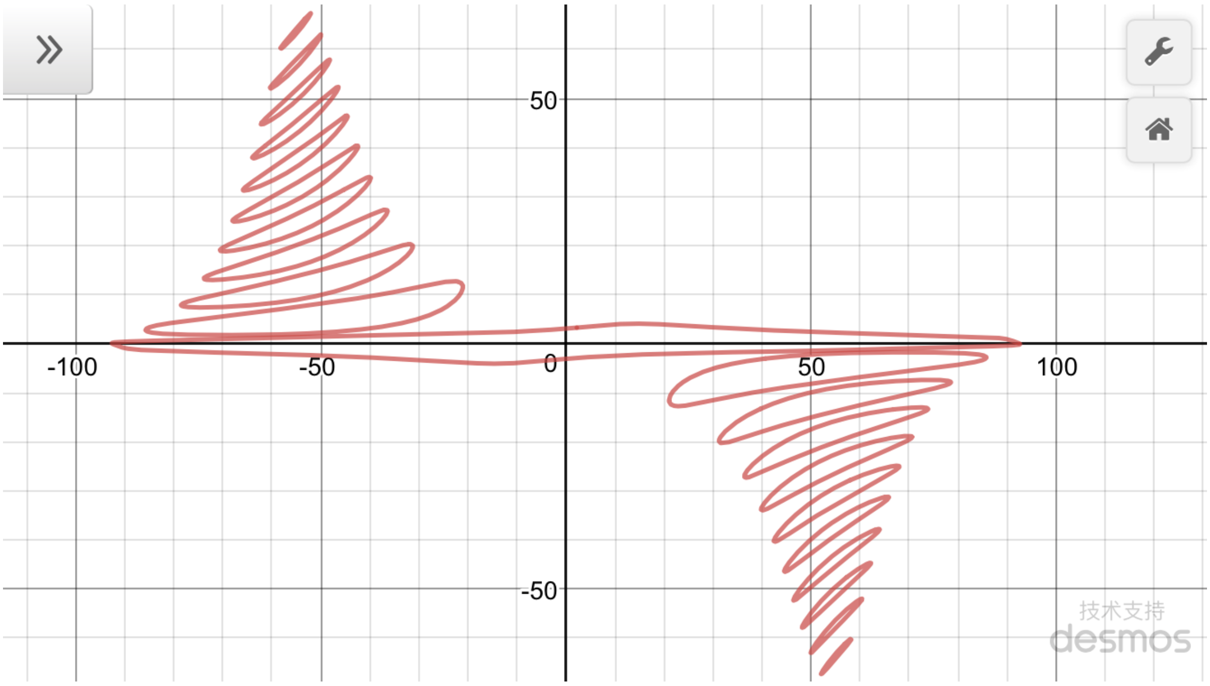


Figure 5h

We extract the useful part of the graph (in the first quadrant) (Figure 5i)

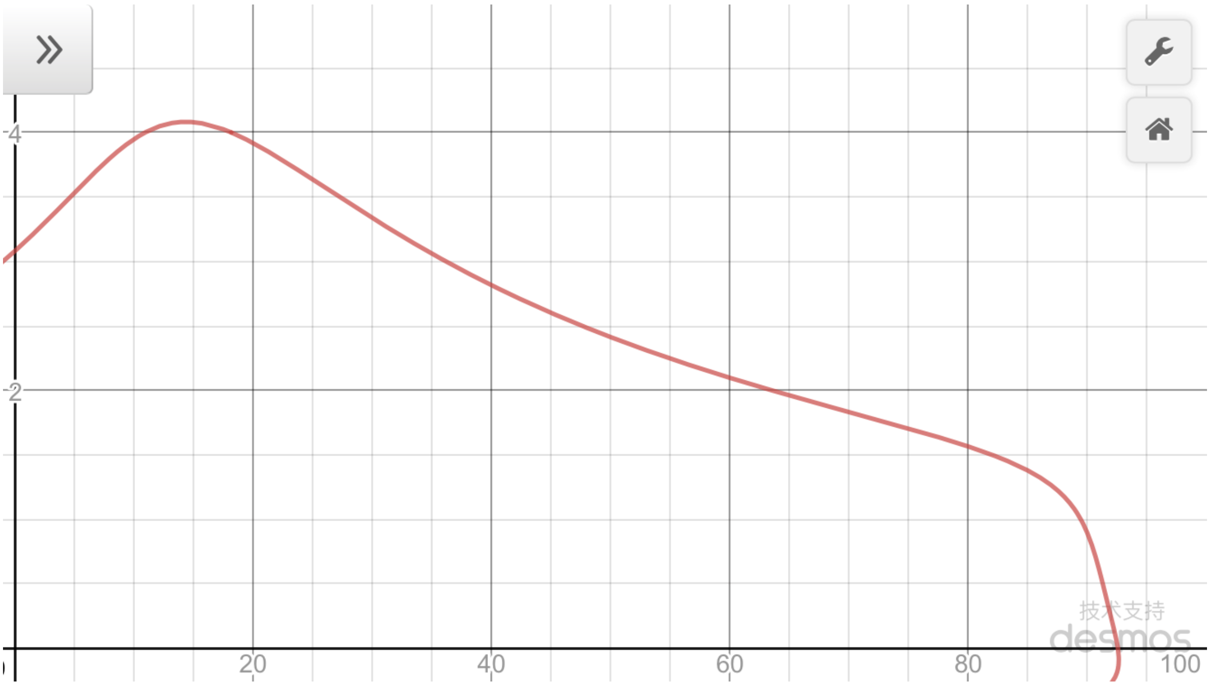


Figure 5i Part of graph in the first quadrant

5.4 Further Research

Figure 5j Breathing on the nose of the paper plane before flying it

Vertically, an air resistance force pointing upwards and gravitational pull pointing downwards are acting on the paper plane, as shown above in Figure 5b. The buoyancy is small and neglectable. However, if the gravitational pull and the air resistance are not acting on the same point, they will make an unbalanced moment of force that will make the plane to spin.

When we look above the wings, they form a triangle. Because the air resistance acts on all parts of the wings, we can assume that it is acting on the geometric center of the triangle, as shown in Figure 5k.

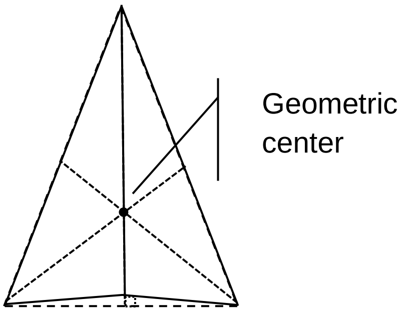


Figure 5k The air resistance acts on the geometric center of the paper plane.

However, does the center of gravity also falls on the geometric center of the triangle? That's not the case. If we see the paper plane horizontally, its shape is as shown in Figure 5l below.

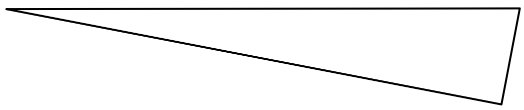


Figure 5l

This shape moves the center of gravity backwards, and causes an unbalanced moment of force that can make the paper plane spin backwards, as shown in Figure 5m. In this case, the best take-off angle cannot be kept.

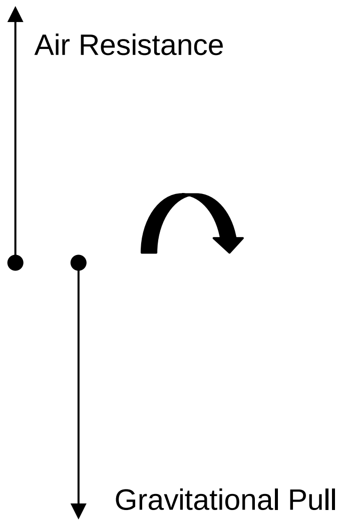


Figure 5m The unbalanced moment of force cause the plane to spin.

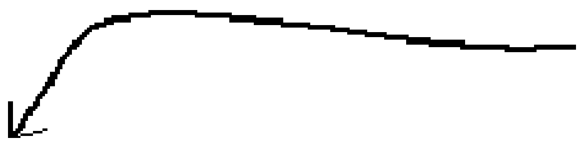


Figure 5n If the plane is too heavy at the back, air resistance will be in front of the gravitational pull, the plane will drop suddenly.

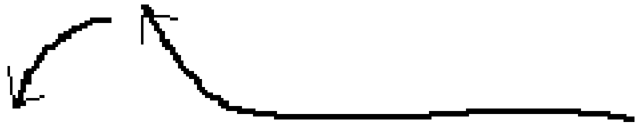
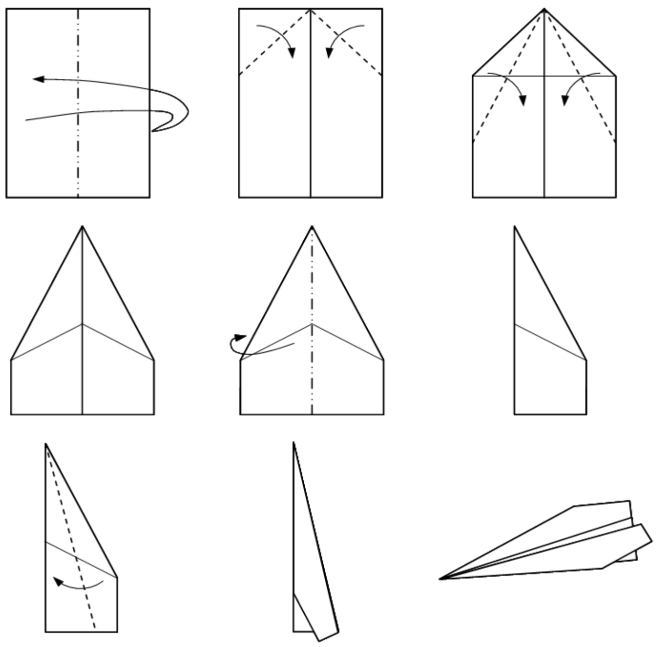


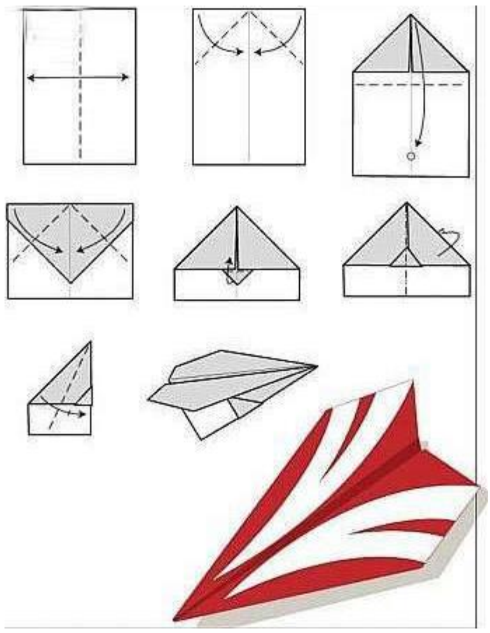
Figure 5o If the plane is too heavy at the front, the plane will go through a sudden climb, losing its speed, and undergoes a sudden drop.

The only way to solve this problem is to increase the weight of the front part of the plane, so that the center of gravity can be move forward, which is also shown in the results of the Taiwanese group that adding two pins at the front of the paper plane will increase its range, bring the position of barycenter of the paper plane closer to the geometric center of the triangular wing, which is the center of lift, hence the moment of force can be more balanced, the paper plane will not increase its angle of attack to deal more air resistance as has stated before in chapter 2 force analysis.

Classical ways of folding paper planes are studied and compared in our process. It turns out that the simplest way of folding a paper plane, as the graph 5p(1b) has shown below, is heavier at the back, the optimum while relatively simple method of folding is shown below in graph 5q.



Graph 5p The simplest way of folding



Graph 5q the improved way of folding

The modified paper plane balances lift center and gravity center because it is folded back in the fourth step, the paper plane ends up shorter and has an extended range.

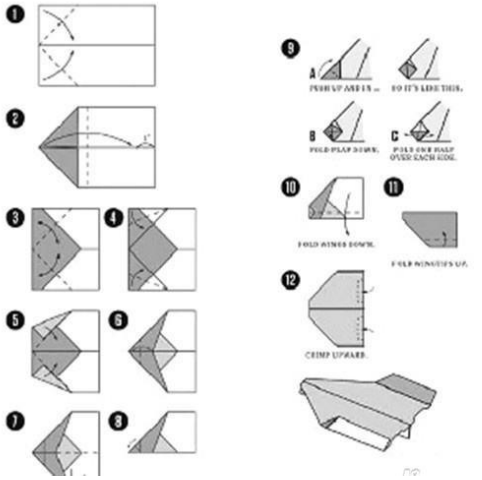


Figure 5r There is also more complex folding way to offer better balance and longer range

Figure 5s There is also folding way with a tail, using the complicated aerodynamics knowledge, the folding way is too complex to be recommended.

6.Conclusion

In a nutshell, how far a paper plane flies depends on its area of wings, its angle of take-off and its position of center of gravity.

For the area of wings, it should be as large as possible.

For the angle of take-off, it should be around 14.5°so that the plane will fly the farthest.

For the center of gravity, it should be at the position of the geometrical center, so that the plane can keep in its stable angle of flying. To ensure this, methods such as breathing on the nose of the plane should be taken to make the front part of the paper plane heavier.

However, in real life, the flying distance of the paper plane involves many other factors. For example, besides the motion in x and y axis, the paper plane itself will turn left or right, or even do aileron rolls, because of its slight asymmetry when folding, or other effects done by air, like wind and vortex. To avoid these from happening, some other structures of paper planes are made.

Y-shaped wings are added to enhance the stability of flying, as shown in Figure 6a, so that the plane will not do aileron rolls.

Figure 6a A paper plane with Y-shaped wings

Sometimes, both tips of the wings are folded so that the paper plane will turn left or right less, as shown in Figure 6b. It also imitates the structure of “wingtip device” of an actual airplane, as shown in Figure 6c, which can reduce wingtip vortex, an effect that increases air resistance.

Figure 6b A paper plane with folded wingtips

Figure 6c A wingtip device of a Cathay Pacific Airways Airbus A350 airplane

7.References

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Annex I

```
#include<iostream>
#include<iomanip>
#include<math.h>
using namespace std;
int main()
{
    double volume=0.000006237;
    float c=1.0;
    float dair=1.205;
    float w=0.0427;
    float m=0.004365;
    float g=9.781;
    float s=0.001
    float v;
    float h;
    float t;
    float fair;
    float fr;
    float a;
    cout<<'s'<<"                    "<<'t'<<"                    "<<'D'<<endl;
    while(s<=0.062)
    {
        v=0.0;
        h=2.0;
        t=0.0;
        cout<<s;
        cout<<"                    ";
        while(h>=0)
        {
            fair=0.5*c*dair*s*v*v;
            fr=m*g-fair-volume*dair*g;
            a=fr/m;
            v+=0.00001*a;
            h-=0.00001*v;
            t+=0.00001;
        }
    cout<<t<<"                    "<<2.0*t<<endl;
    s+=0.001;
    }
}
```

Definitions:

dair: air density

w: weight

m: mass

g: acceleration due to gravity

s: area of wings

v: vertical velocity

h: current height

t: current time

f air: drag

fr: vertical resultant force

a: current vertical acceleration