SMTP (Math)

Square sum problem

Yu Wenhao 3S1 (29) Leader

Koh Shao Bing 3S1 (11)

Sean Tan Liyu 3S1 (21)

Hwa Chong Institution

(High School)

1 Introduction

The square sum problem refers to a line of integers consisting of all natural numbers from 1 to n such that the sum of every two adjacent numbers sum up to a perfect square, and finding what are the values of n in which this construction is possible. There have been some proof found for pairing integers from 1 to n where n is an even number such that each pair has a square sum, but not for the square sum problem; and we intend to build upon their research and attempt to prove the existence or the non-existence of such constructions, and the required conditions in which such constructions occur.

We will also like to investigate other extensions to this problem such as applying the rule for the square sum problem to other famous sequences such as the the fibonacci sequence or powers of 2.

1.1 Objectives

1. To investigate the solution of the square sum paired partition problem
2. To investigate the code which generate values of n for the square sum problem
3. To find an analytic solution for the square sum problem
4. To investigate further extensions of the square sum problem

1.2 Research Questions

1. How do we pair up integers in the range 1 to $2n$, where n is an integer, such that the sum of each pair adds up to a square number?
2. How does the algorithmic solution of the square sum problem work?
3. Is there an analytic solution to the square-sum problem?
4. Are there solutions for the square sum problem where the sum of adjacent numbers are of other types, such as cube numbers or fibonacci numbers?

1.3 Fields of math

1. Number Theory
2. Graph Theory
3. Algebra

1.4 Terminology

Hamiltonian Path	Graph path between two vertices of a graph that visits each vertex exactly once
Graph	representation of a set of points, in which points are connected by lines between each pair of points.
Edges	lines connecting points
Vertices	points in the graph

Square Sum problem	The problem of how to arrange numbers from the range 1 to n, where n is a positive integer, into a line such that the sum of every two adjacent numbers are square numbers.
Square Sum paired partition problem	The problem of how to pair numbers from the range 1 to n, where n is a positive integer, such that the sum of the two numbers in each pair is a square number
Induction	A method of proof in which a statement is proved for one step in a process, and it is shown that if the statement holds for that step, it holds for the next.

2 Literature Review

2.1 Square-sum paired partition

Gordon Hamilton (2015) wrote a research paper on square-sum paired partition proof and an overview of methods related to solving square-sum paired partition. The research paper discusses methods using number induction to solve the problem and a more brute force way of rainbow pairing to solve it. This paper is highly relevant to this project as our project concerns adjacent numbers adding to a perfect square and this research paper talks about splitting numbers from 1 to n (where n is even) into pairs such that each pair add up to a perfect square .

However, as mentioned above, this research paper only covers about splitting the sequence into pairs while our project is looking into finding a solution for adjacent numbers to add up to a perfect square. Hence, we will aim to find a solution for the problem in which the numbers 1 to n are arranged such that adjacent numbers add up to perfect squares.

2.2 Hamiltonian Path

In graph theory, a “graph” refers to a representation of a set of points, in which points are connected by lines between each pair of points. The lines connecting points are known are “edges” and the points in the graph are known as “vertices”.

A Hamiltonian path is a graph path between the vertices of the graph that visits each vertex exactly once. The square sum problem is an example of a Hamiltonian path, viewing every integers from 1 to n as a point on the graph; a valid construction occurs when there is a “path” from one number to another, passing through all numbers but only passing each number only once.

2.3 Algorithmic solution for square sum problem

While no research have be done to solve the square-sum problem using mathematical concepts, there have been various algorithmic solutions which involves using each integer and trying to find the hamiltonian path for each integer. Hence, we will be studying these solutions and using it to find patterns and use mathematics to solve the square sum problem.

3 Methodology

1. Read up on resources related to square-sum problem

2. Thoroughly understand and apply the square-sum paired partition proof in our project through the reading of relevant research papers and other sources
3. Thoroughly understand the square sum algorithmic solution through the reading of relevant information and source code stated in the Literature Review
4. Attempt to find a square sum analytical by first using the code to generate values of n and subsequently searching for a pattern and attempt to generalise the solution
5. Attempt to solve the square sum problem for adjacent numbers adding up to other types of numbers such as fibonacci numbers by using both the algorithmic and analytic (if possible) solution of the square sum problem.

4 Findings

4.1 The square sum paired partition problem proof

The square sum paired partition problem was to pair numbers from 1 to $2n$ such that every pair added up to a square number. The proof of the square sum paired partition problem was done by Gordon Hamiltonian.

The proof is done through strong induction, using $n \leq 30$ (i.e. $2n \leq 60$) as base cases. Refer to Appendix A for base cases for $n \leq 30$. Assuming there exists a pairing for some $2m$, we will pair $2m+1$ to $2n$, $2m+2$ to $2n-1$ and so on to achieve a new pairing. So hence we need to have some conditions for our induction.

1. $2m+1+2n = k^2$
2. $m < n$, since we are building a new pairing for $2n$ with $2m$ as a base case
3. $m \geq 12$, to avoid the base cases that do not work (Refer to Appendix A)

Since k^2 is obviously odd, it must take the form of

$$(2k_1 + 1)^2 = 4k_1^2 + 4k_1 + 1$$

By substitution into the 1st condition, we have

$$m + n = 2k_1^2 + 2k_1$$

Since $m < n$,

$$n \geq k_1^2 + k_1 + 1$$

Since $m \geq 12$,

$$n \leq 2k_1^2 + 2k_1 - 12$$

To ensure the induction continues, we need to ensure that the upper bound of the previous case overlaps with the lower bound of the next case, hence we have

$$(k_1 + 1)^2 + (k_1 + 1) + 1 \leq 2k_1^2 + 2k_1 - 12$$

It is then simple to show that this inequality will hold for all $k_1 \geq 5$, i.e. for all $n \geq 5^2 + 5 + 1$.

Hence we can see that the induction will hold for all $n \geq 31$.

Hence the induction is complete

4.2 Algorithmic solution of the square sum problem:

The algorithmic solution of the square sum problem is a solution that uses brute force to find a solution instead of using mathematical concepts. However, it can still be useful for developing the analytical solution, the one that utilises mathematical concepts to solve the square sum problem. Refer to Appendix B for the code, written in C++ Language.

4.2.1 Algorithmic Solution

Following is an explanation of the code.

Two lists are initialised, l and m. The ith element of each list is denoted as l_i and m_i respectively. l_2 to l_n and m_2 to m_n are all initialised to be 0. l_1 and m_1 are both defined as 1. One of the following actions would be chosen based on the conditions. This step is referred to in the future as the choosing of action, considering the element in l that needs to be updated next. Actions that are listed first are considered first.

- If the list has n elements, terminate the code. The solution is the current list l
- If the list has no elements within, increment m_1 by 1 and set $l_1 = m_1$. Repeat the choosing of action by considering l_2
- Let the leftmost element with a value of 0 be l_i. Set

$$l_i = (\sqrt{m_i + l_{i-1}} + 1)^2 - l_{i-1}$$

  - If $l_i \le n$, Set $m_i = l_i$. Repeat the choosing of actions, considering l_{i+1}
  - If $l_i > n$, Set $m_i = 0$, $l_{i-1} = 0$. Repeat the choosing of actions, considering l_{i-1}

4.2.2 Explanation of Algorithmic Solution

The list l acts as the storage for the solution, while the list m acts as the storage for past solutions. The third action is the main action amongst the three actions, while the first two are to check for the completion of the solution and a special case, when the list is empty, respectively. The third action checks whether there is a number that is within the boundaries of n that can be added to the previous number in list l. If there exists such a number, the first scenario of the third action is activated, and the number is stored in list l. If there does not exist such a number, the second scenario of the third action is activated, and the previous number is removed. This process will be repeated until the list l has n elements, which is when the list is full.

4.3 Analytical solution of the square sum problem:

While we have not been able to find an analytical solution for the square sum problem, using probability, we are able to determine that for sufficiently large values of n, there is a near 100% chance of finding a solution for the value of n.

This probability version of the proof uses induction from one case to next, from $n-1$ to n. In this proof, two numbers i and j are randomly chosen between 1 and $n-1$, with $i<j$. Note that there are $\binom{n-1}{2}$ ways to choose i and j.

Flip all the numbers that are between the ith index and jth index inclusive, and insert the number n between the number with $(i-1)$th index and the jth index. The elements of the list would be labelled as l_{index}.

1	...	i-1		(n)		j	j-1	...	i+1	i		j+1	...	n-1

This is the case for n, but for such a sequence to be valid, $l_{i-1}+n$, $n+j$, $i+j+1$ must all be perfect squares.

The chance that one of the three terms is a perfect square is roughly $\frac{\sqrt{n}}{n}$, as there is roughly $\sqrt{n}$ numbers below n that when added to n forms a perfect square, over the total of n numbers that are in the list.

The probability for all three terms to be perfect squares is

$$\left(\frac{\sqrt{n}}{n}\right)^{3} = n^{-\frac{3}{2}}$$

Since there are $\binom{n-1}{2} \approx \frac{n^2}{2}$ combinations for i and j, the probability that a scenario where all three terms are perfect squares is not possible is

$$(1 - n^{-\frac{3}{2}})^{\frac{n^2}{2}}$$

We can observe that this equation is strictly decreasing when $n > 1$, as the value of $(1 - n^{-\frac{3}{2}})$ would decrease further less than one and the value of $\frac{n^2}{2}$ increasing.

When $n \geq 85$, the chance of such a construction being possible is larger than 99.9%.

By searching from 25 to 2^{20}, the highest number that is unable to be constructed with such a method is when $n = 6109$, at which point the chance of such a construction not being possible is 1×10^{-17}. Hence, the reliability of this method increases as the value of n increases, and this proof can be very accurate when the value of n is very large.

4.4 Variation of the square sum problem:

4.4.1 Fibonacci sequence:

Instead of having adjacent numbers add up to a square number, we have the sum of every pair of adjacent number to be an integer in the Fibonacci Sequence.

The Fibonacci Sequence follows the recurrence relation,

$$F_n = F_{n-1} + F_{n-2}$$

$$F_1 = F_2 = 1$$

In our proof, it is essential to remember that for a hamiltonian path to exist in a graph, there can only be a maximum of 2 vertices which are only connected by 1 edge. Let each vertice of the graph represent an integer and an edge connecting two vertices representing that their sum is a Fibonacci number.

Here we will start by considering the upper bound. Let's assume that there are n nodes, numbered from 1 to n.

Consider n to be a Fibonacci number, say F_j.

Since F_j cannot be added with another natural number to achieve a sum of itself, F_j. We consider the next lowest target, F_{j+1}, which works as shown.

$$F_j + F_{j-1} = F_{j+1}$$

However, to reach the next smallest Fibonacci number, F_{j+2},

$$F_j + F_{j+1} = F_{j+2}$$

This indicates that F_j, must be paired with F_{j+1} to achieve the next smallest target. However, since this construction must hold within the condition that F_j is the maximum number. Since the Fibonacci sequence is strictly increasing, F_{j+1} must be larger than F_j and hence cannot exist in the construction.

A similar argument is used for $F_j + 1$.

At this point, we realise that from $F_j + 2$, onwards, there will exist more than 2 vertices which only have 1 edge, making a hamiltonian path impossible.

We notice that the number of vertices with only 1 edge returns to 2 at $F_{j+1} - 2$, in which F_j and $F_j + 1$ are the two numbers with 1 edge.

Hence we can say that all possible values of n in which there exists a solution must take the form of $F_j - 2$, $F_j - 1$, F_j or $F_j + 1$.

By the nature of the recurrence pattern of the Fibonacci sequence, we notice that the Fibonacci sequence follows an Odd-Odd-Even cycle every 3 numbers. Keeping in mind that we can only have a maximum of 2 vertices with only 1 connected edge.

Firstly, for every even Fibonacci number, which occurs at F_{3n}, then half of F_{3n} also exists in the sequence.

$$\frac{F_{3n}}{2} + \frac{F_{3n}}{2} = F_{3n}$$

However, the same number cannot appear twice, which means that $\frac{F_{3n}}{2}$ will only have 1 edge until, $F_{3n-1} + \frac{F_{3n}}{2} > F_{3n}$, is the pair, with the target F_{3n+1}

$$(F_{3n+1}) - (\frac{F_{3n}}{2}) = (F_{3n-1} + \frac{F_{3n}}{2})$$

Hence at $n = F_{3n}$, both $\frac{F_{3n}}{2}$ and F_{3n} have 1 edge.

And at $n = F_{3n} - 1$, both $\frac{F_{3n}}{2}$ and F_{3n-1} have 1 edge.

This means that the cases for $n = F_{3n} - 2,\ F_{3n} + 1$ cannot have a solution. Given that there would then be 3 vertices with only 1 edge.

Next, for every 2nd Odd Fibonacci number in the 3 number cycle, say F_{3n-1}.

By the nature of the recurrence relation of the Fibonacci sequence, we notice that,

$$F_{3n-1} > \frac{F_{3n}}{2}.$$

Hence we realise that when $n = F_{3n-1} - 1$, both F_{3n-2} and $\frac{F_{3n}}{2}$ have 1 edge

And when $n = F_{3n-1}$, both F_{3n-1} and $\frac{F_{3n}}{2}$ have 1 edge.

Similarly, we can hence rule out cases of $n = F_{3n-1} - 2,\ F_{3n-1} + 1$ having possible solutions too.

Lastly, we will look at every 1st Odd Fibonacci Number in every cycle of 3, say F_{3n+1}.

We aim to prove that $n = F_{3n+1} - 2,\ F_{3n+1} + 1$ cannot have any solutions.

Assume that there exists such solutions.

Hence for $n = F_{3n+1} - 2$, both F_{3n} and $F_{3n} + 1$ have exactly 1 edge

And for $n = F_{3n+1} + 1$, both F_{3n+1} and $F_{3n+1} + 1$ have exactly 1 edge

Hence we can conclude that the above mentioned two integers with only 1 edge must be at each end of the hamiltonian path.

First, given $n = F_{3n+1}$, all k in the sequence in which $k \geq F_{3n-1}$, have either 2 or less edges.

Consider $\frac{F_{3n}}{2}$.

We realise that $\frac{F_{3n}}{2}$ can be paired with $\frac{F_{3n-3}}{2}$ or $\frac{F_{3n}}{2} + F_{3n-1}$ only

$$\frac{F_{3n}}{2} + \frac{F_{3n-3}}{2} = \frac{F_{3n-1}+F_{3n-2}+F_{3n-3}}{2} = F_{3n-1}$$

$$\frac{F_{3n}}{2} + \left(\frac{F_{3n}}{2} + F_{3n-1}\right) = F_{3n+1}$$

Next consider $\frac{F_{3n}}{2} + F_{3n-1}$. As stated above, this integer can only have 2 edges

We realise that it can be paired with $\frac{F_{3n}}{2}$, as stated above, and $\frac{3F_{3n}}{2}$

$$\left(\frac{F_{3n}}{2} + F_{3n-1}\right) + \frac{F_{3n}}{2} = F_{3n+1}$$

$$(\frac{F_{3n}}{2} + F_{3n-1}) + \frac{3F_{3n}}{2} = F_{3n+2}$$

Once again, consider $\frac{3F_{3n}}{2}$,

We realise that it can be paired with $\frac{F_{3n}}{2} + F_{3n-1}$, as stated above, and $F_{3n+1} - \frac{3F_{3n}}{2}$

$$\frac{3F_{3n}}{2} + (\frac{F_{3n}}{2} + F_{3n-1}) = F_{3n+2}$$

$$\frac{3F_{3n}}{2} + (F_{3n+1} - \frac{3F_{3n}}{2}) = F_{3n+1}$$

Here we realise that

$$F_{3n+1} - \frac{3F_{3n}}{2} = \frac{F_{3n-3}}{2}$$

Hence we conclude that the four above mentioned integers, $\frac{F_{3n}}{2}$, $\frac{F_{3n-3}}{2}$, $\frac{3F_{3n}}{2}$ and $\frac{F_{3n}}{2} + F_{3n-1}$, will form a cycle like so. And since $\frac{F_{3n}}{2}$, $\frac{3F_{3n}}{2}$ and $\frac{F_{3n}}{2} + F_{3n-1}$ only have 2 connections, we will arrive at a situation depicted below. Hence a hamiltonian path cannot occur.

Thus $n = F_{3n+1} - 2,\ F_{3n+1} + 1$ have no solution.

Overall, we can conclude that all possible solutions must take the form of $n = F_j ,F_j - 1$

Now onto our prove to show that a hamiltonian path exist for all fibonacci numbers. Do take note that since the fibonacci number will be at the end of the hamiltonian path, should there exist a hamiltonian path for all fibonacci numbers, one could take away the fibonacci number at the end to form a hamiltonian path with Fj-1.

Using the sequence of 34, there exist a hamiltonian path which is 17-4-30-25-9-12-22-33-1-20-14-7-27-28-6-15-19-2-32-23-11-10-24-31-3-18-16-5-29-26-8-13-21-34

Between the difference of each adjacent numbers in this sequence there is a presence of numbers from 1 to 20, followed by 22,24,26,28,30 and 32. Hence the next fibonacci number which is 55. By slotting the pairs of 35-54, 36-53, 37-52, 38-51, 39-50, 40-49, 41-48, 42-47, 43-46 and 44-45 between the pairs 1-20, 2-19, 3-18, 4-17, 5-15, 6-14, 7-13, 8-12, 9-11 respectively, and adding 55 to the end of the path, the hamiltonian path of Since the difference of 1,3,5…..19 is preserved when we insert the above pairs. New differences is added as the difference between Fn-k We can further prove this works for all fibonacci numbers above 34. By splitting this case into two cases, first case which the difference between the fibonacci numbers is odd and the other case where the difference is even

Hence we have proven that for all fn and fn-1 that is above 33, a hamiltonian path exists

4.4.2 Powers of 2 minus 1:

Instead of adjacent numbers adding up to a square number, we have the sum of every pair of adjacent numbers add up to a power of 2 minus 1. We will use proof by contradiction.

Assume that there exists a solution for some n.

Consider the greatest 2^a and $2^a - 1$ such that,

$$2^a,\ 2^a - 1 \leq n.$$

We notice that both 2^a and $2^a - 1$ cannot be paired up with another natural number to have the sum of $2^a - 1$.

And if the target sum is $2^{a+2} - 1$,

$$(2^{a+2} - 1) - (2^a) = 3(2^a) - 1$$

$$(2^{a+2} - 1) - (2^a - 1) = 3(2^a)$$

The two integers will have to be paired with integers $3(2^a) - 1$ and $3(2^a)$ respectively, which are obvious larger than 2^{a+1}

Therefore by the conditions that 2^a and $2^a - 1$ are the largest such numbers that are in the range of possible integers, $3(2^a) - 1$ and $3(2^a)$ cannot exist in the set of numbers

Hence these two integers cannot be paired up with the two given numbers.

Hence we can conclude that 2^a and $2^a - 1$ can only be paired with each other.

$$(2^a) + (2^a - 1) = 2^{a+1} - 1$$

When we consider the graph of natural numbers from 1 to n, with each vertice representing an integer and an edge connecting two vertices representing their sum to be of the form $2^k - 1$.

We realise that there will always exist two such 2^a and $2^a - 1$ in which are only connected to each other.

Hence a hamiltonian path cannot exist in such a graph.

The only exception is when $a = 1$, with the only two numbers being 1 and 2. In this case, the hamiltonian path exists with 1 being adjacent to 2 and their sum $2^2 - 1 = 3$.

5 Conclusion

The square sum paired partition problem can be solved through strong mathematical induction

For the variations of the square sum problem, for fibonacci, we have proven that a hamiltonian path exists for all $n = F_j , F_j - 1$. For powers of 2 minus 1, we have proven a hamiltonian path does not exist for all values of n except for n=2

For the square sum problem, we were unsuccessful in solving it, but managed to deduce the probability of such a hamiltonian path to exist for n as we

6 Timeline

Time	Work to be done
Term 1, Week 6 to 7	Brainstorm for project idea
Term 1, Week 8 to 9	Complete project proposal
March Holidays	Start and complete powerpoint slides
Term 2, Week 1 to 2	Rehearse for project
Term 2, Week 3	Proposal Evaluation
Term 2, Week 4 - 10	Prepare for Mid-term Evaluation
Term 3, Week 1 and 2	Rehearse for Mid-term Evaluation
Term 3, Week 3	Mid-term Evaluation
Term 3, Week 4 - 7	Prepare for Final Evaluation
Term 3, Week 8	Final Evaluation

Term 3, Week 9 and 10	Wrap up project

7 References

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8 Acknowledgements

We would like to express our deepest gratitude to our project mentor, Mr Chen Zenghui, for patiently mentoring us through this entire project, and lending a hand whenever we needed help.

In addition, we would also like to thank our school, Hwa Chong Institution, for providing us with this golden opportunity to conduct our research on this project.

Appendix A:

n=4: [1-8, 2-7, 3-6, 4-5]

n=7: [1-8, 2-14, 3-13, 4-12, 5-11, 6-10, 7-9]

n=8: [1-8, 2-7, 3-6, 4-5, 9-16, 10-15, 11-14, 12-13]

n=9: [1-15, 2-14, 3-13, 4-12, 5-11, 6-10, 7-18, 8-17, 9-16]

n=12: [1-24, 2-23, 3-22, 4-21, 5-20, 6-19, 7-18, 8-17, 9-16, 10-15, 11-14, 12-13]

n=13: [1-15, 2-23, 3-22 4-21, 5-20, 6-19, 7-18, 8-17, 9-16, 10-26, 11-14, 12-13, 24-25]

n=14: [1-15, 2-14, 3-22, 4-12, 5-11, 6-19, 7-18, 8-17, 9-27, 10-26, 13-23, 16,20, 21-28, 24-25]

n=15: [1-15, 2-14, 3-13, 4-12, 5-11, 6-10, 7-18, 8-17, 9-16, 19-30, 20-29, 21-28, 22-27, 23-26, 24-25]

n=16: [1-8, 2-7, 3-6, 4-5, 9-16, 10-15, 11-14, 12-13, 17-32, 18-31, 19-30, 20-29, 21-28, 22-27, 23-26, 24-25]

n=17: [1-8, 2-14, 3-13, 4-12, 5-11, 6-10, 7-9, 15-34, 16-33, 17-32, 18-31, 19-30, 20-29, 21-28, 22-27, 23-26, 24-25]

n=18: [1-15, 2-23, 3-33, 4-32, 5-20, 6-19, 7-18, 8-17, 9-16, 10-26, 11-25, 12-24, 13-36, 14-35, 18-31, 21-28, 22-27, 30-34]

n=19: [1-8, 2-14, 3-22, 4-5, 6-19, 7-18, 9-27, 10-15, 11-38, 12-37, 13-36, 16-20, 17-32, 21-28, 23-26, 24-25, 29-35, 30-34, 31-33] n=20: [1-8, 2-7, 3-6, 4-5, 9-40, 10-39, 11-38, 12-37, 13-36, 14-35, 15-34, 16-33, 17-32, 18-31, 19-30, 20-29, 21-28, 22-27, 23-26, 24-25]

n=21: [1-8, 2-14, 3-22, 4-5, 6-19, 7-18, 9-27, 10-15, 11-38, 12-37, 13-36, 16-20, 17-32, 21-28, 23-26, 24-25, 29-35, 30-34, 31-33, 39-42, 40-41]

n=22: [1-15, 2-23, 3-33, 4-32, 5-20, 6-19, 7-29, 8-17, 9-16 10-26, 11-25, 12-24, 13-36, 14-35, 18-31, 21-28, 22-27, 30-34, 37-44, 38-43, 39-42, 40-41]

n=23: [1-8, 2-14, 3-13, 4-12, 5-11, 6-10, 7-9, 15-34, 16-33, 17-32, 18-31, 19-30, 20-29, 21-28, 22-27, 23-26, 24-25, 35-46, 36-45, 37-44, 38-43, 39-42, 40-41]

n=24: [1-8, 2-7, 3-6, 4-5, 9-16, 10-15, 11-14, 12-13, 17-32, 18-31, 19-30, 20-29, 21-28, 22-27, 23-26, 24-25, 33-48, 34-47, 35-46, 36-45, 37-44, 38-43, 39-42, 40-41]

n=25: [1-15, 2-14, 3-13, 4-12, 5-11, 6-10, 7-18, 8-17, 9-16, 19-30, 20-29, 21-28, 22-27, 23-26, 24-25, 31-50, 32-49, 33-48, 34-47, 35-46, 36-45, 37-44, 38-43, 39-42, 40-4]

n=26: [1-15, 2-14, 3-22, 4-12, 5-11, 6-19, 7-18, 8-17, 9-27, 10-26, 13-23, 16,20, 21-28, 24-25, 29-52, 30-51, 31-50, 32-49, 33-38, 34-47, 35-46, 36-45, 37-44, 38-43, 39-42, 40-41]

n=27: [1-15, 2-23, 3-22 4-21, 5-20, 6-19, 7-18, 8-17, 9-16, 10-26, 11-14, 12-13, 24-25, 27-54, 28-53, 29-52, 30-51, 31-50, 32-49, 33-38, 34-47, 35-46, 36-45, 37-44, 38-43, 39-42, 40-41]

n=28: [1-24, 2-23, 3-22, 4-21, 5-20, 6-19, 7-18, 8-17, 9-16, 10-15, 11-14, 12-13, 25-56, 26-55, 27-54, 28-53, 29-52, 30-51, 31-50, 32-49, 33-38, 34-47, 35-46, 36-45, 37-44, 38-43, 39-42, 40-41]

n=29: [1-15, 2-23, 3-13, 4-12, 5-20, 6-19, 7-18, 8-41, 9-40, 10-26, 11-38, 14-22, 16-33, 17-32, 21-43, 24-57, 25-39, 27-37, 28-36, 29-35, 30-34, 31-50, 42-58, 44-56, 45-55, 46-54, 47-53, 48-52, 49-51] n=30: [1-8, 2-7, 3-33, 4-5, 6-43, 9-27, 10-54, 11-38, 12-37, 13-36, 14-35, 15-34, 16-48, 17-47, 18-46, 19-45, 20-44, 21-60, 22-59, 23-58, 24-57, 25-56, 26-55, 28-53, 29-52, 30-51, 31-50, 32-49, 39-42, 40-41]

Appendix B:

The following is the code in C++ Language

```
bool procedure(bitset <N+1> used, int start){
    if (chain.empty()){
        chain.emplace_back(start);
        used[start]=1;
    }
    bitset <N+1> _def; _def.set();
    if (used == _def){
        output(chain);
        return 1;
    }
    unsigned short last = chain.back(), rlast = sqrt(last);
    for (unsigned short i = rlast + 1; i <= N; ++i){
        unsigned short difference = i * i - last;
        if (difference < 1 || difference > N)break;
        if (used[difference]) continue;
        used[difference] = 1; // Used
        chain.emplace_back(difference);
        if (!procedure(used, start)) {
            used[difference] = 0; // Undo
            chain.pop_back();
        }
    }
    return 0; // Cannot
```