SMTP(Math) Written Report

The Control Room Riddle

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Terminology

Edge	A line that interconnects two nodes. Represents the connection between two rooms in this project.
Node/Vertex	The common endpoint of two or more lines segments. Represents a room in this project.
Degree	Number of edges which is in touch with a node
Brute Force Approach	Checking each and every possible path to a problem and deducing which is correct
Degree Sum Formula/Handshaking Lemma	A formula which states that in any undirected graph, a graph has to 1) Have an even number of vertices

	with odd degree 2) Sum of degree has to always be even

Variables

n	The highest possible floor the control room can be on / Total no. of rooms
m	Degree of non-control room
k	Degree of control room
p	Sum of Degree
c	No. of control room

1 Introduction

Control Room Riddle is a popular riddle which is featured by TED-Ed. An agent, which belong to a spy organisation has to infiltrate into an enemy headquarters, which is a pyramidal building, consisting of 10 floors. Contrary to a normal building, in the pyramid, the 1st floor is the top floor and the 10th floor is at the bottom. The agent has to investigate and find out which floor the control room is on.

The following information if provided:

- The floor number represents the number of rooms the floor has

  (1st floor has 1 room, 2nd floor has 2 rooms, xth floor has x rooms)
- Each room is connected to 3 other rooms (except the control room)
- Control room is connected to only 1 room
- The rooms are connected within the same floor

One thing he knows for sure is that the control room is on the highest possible floor, which is the smallest floor number, which fulfills all the conditions stated above. Hence, the agent has to find out the highest possible floor the control room is situated on.

In our project, we let

- Control room be represented as CR
- Non-control rooms to be represented as alphabets, A-Z
- The pyramid consist of an infinite number of floors

(We only considered the possibility of simple, undirected graphs)

(Original Solution)

The original solution states that the control room is on the **6th floor.**

- Control rooms as CR
- Non-control rooms as alphabets A-Z
- Rooms are represented as nodes
- Connections between rooms are represented as edges

1. Since each floor is connected to 3 other rooms. There must be a total of 4 rooms. Thus, floors 1, 2, 3 are ruled out

2. On 4th floor, where there are 3 normal rooms and 1 control room, where the normal rooms will be denoted by A, B and C. (Shown in Fig 1.1)

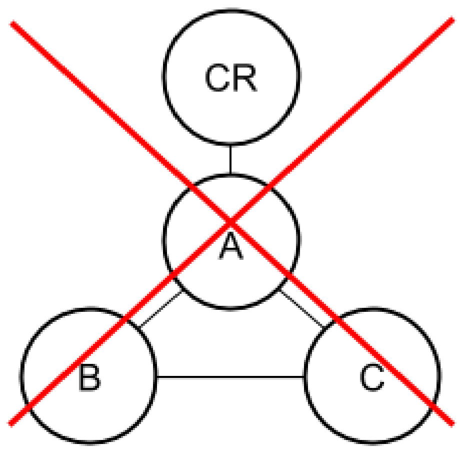


Fig 1.1 Graph of possible floor plan of Floor 4

Graph of g1 = (v1, e1),

v1 = {A, B, C, CR}, e1 = { {CR,A} {A,B} {A,C} {B,C} }

Rooms B and C only have a degree of 2. Thus, Floor 4 is ruled out.

3. In odd-numbered floors, like floor 5, floor 7, floor 9, the result will be a graph, where the sum of degree of all nodes is odd, this contradicts the Handshaking Lemma, and hence cannot be the solution for the riddle. Thus, all odd-numbered floors are ruled out.

4. Using the Brute Force Approach, it is observed that floor 6 fulfills all conditions. (Shown in Fig 1.2)

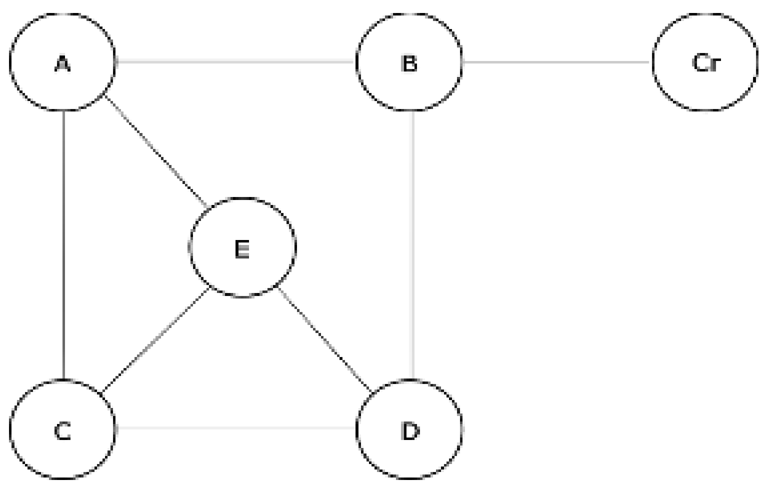


Fig 1.2 Graph Of floor plan of Floor 6

Graph of g2 = (v2, e2),

v2 = {A, B, C, D, E, CR},

e2 = { {A, B}, {A, E}, {A, C}, {B, CR}, {B, D}, {C, E}, {C, D}, {D, E} }

Hence, floor 6 is the solution.

1.1 Rationale

This riddle is spellbinding and fascinating. We were deeply intrigued by how complex questions can be minced and broken down into simple undirected graphs. A question struck us. How will changing the conditions affect the highest possible floor the control room can be on? This led us to coming up with the three different research questions. We hope that by doing this, we can gain a deeper insight into how this riddle works as a whole.

1.2 Objectives

1) To solve for the highest possible floor a control room can be on by changing different variables in the original problem
2) Generalise the original problem and investigate if there are any restrictions or limitations

1.3 Research Questions

1) What is the highest possible floor for the control room to be on if each room (except for the control room) is connected to ***m*** other rooms?

2) How will the solution change if each control room is connected to ***k*** other rooms?

3) How will the solution change if ***c*** more control rooms are introduced?

1.4 Field Of Mathematics

1) **Graph Theory**
    - Rooms will be represented as nodes (vertices)
    - Connectors between the rooms are represented as edges (links)

This allows us to break down the riddle into a simple undirected graph diagram to allow us to deduce whether the current floor fulfills the conditions of the riddle and to check for any oversights and errors in our workings.

2) Algebra

We use algebra to

- Express degree of non-control room as m
- Express degree of control room as k
- Express number of control rooms as c
- Express highest possible floor the control room can be on using n
- Express the sum of degree of all vertices as p
- Form the equation $m(n - c) + kc = p$ using all the variables

This allows us to express our research questions in a way that others and ourselves can interpret more clearly.

We are also able to present the relationship between the highest floor the control room can be on and the condition changed in a straightforward approach using the equation.

It is used to further simplify the problem and by making use of the Handshaking Lemma along with Algebra, we are able to solve our research questions.

2. Literature Review

2.1 Graph Theory

First used in Seven Bridges of Königsberg problem (Fig 2.1). It was then proven impossible by Leonhard Euler (1736)

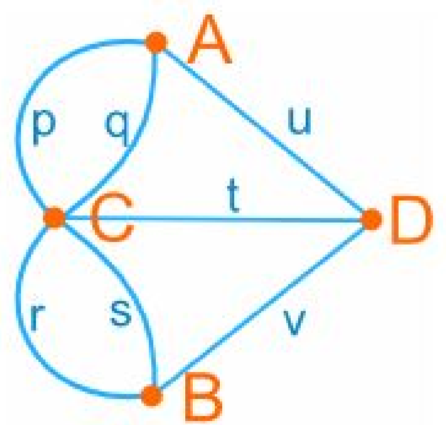


Fig 2.1 Seven Bridges of Königsberg problem

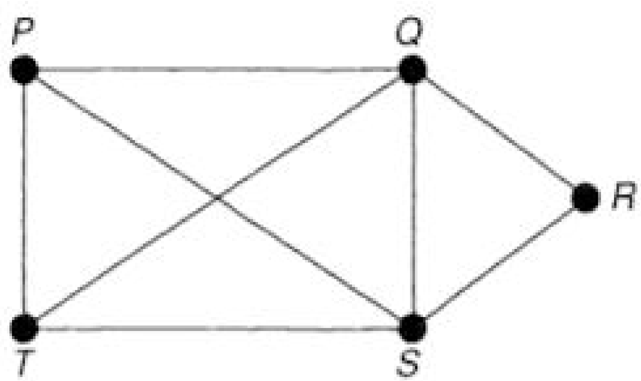


Fig 2.2: A Graph

In this case, P, Q, R, S and R are nodes and the lines between 2 nodes are the edges. This graph can be used in a soccer tournament. In a soccer tournament, the nodes can be represented as soccer.teams. The edges depict which teams have played against each other. We can infer from the graph that Team P has played against Team Q, T and S. The number of edges represents the number of soccer matches played in total (Robin J Wilson, 1970)

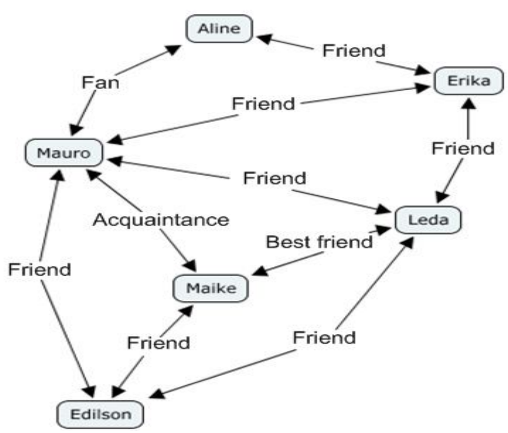


Fig 2.3 Acquaintance network

Graph theory can also be used in an acquaintance network. Node represents 1 person while an edge represents the relationship between 2 people (Richard J Trudeau,1993).

Similar to these two examples, graph theory can be applied in our project. The rooms (Both control and non-control rooms) are represented as nodes while the connection between the rooms are represented as edges.

2.2 Handshaking Lemma (Degree Sum formula)

$$\sum_{v \in V} \deg v = 2|E|$$

Fig 2.3: Handshaking Lemma

The formula is given by Fig 2.3 where V represents the vertex set and E representing the edge set. The sum is even when there is an even number of odd terms. When there is an odd number of vertices with odd degree, the sum of degree will be odd since an odd integer multiplied by another yields an odd integer. Thus, there must be an even number of vertices with odd degree in order for the sum of degree to be even. Hence, the formula states that every finite undirected graph has an even number of vertices with odd degree.

This is proven by Leonhard Euler (1736) in his famous paper on the Seven Bridges of Königsberg that began the study of graph theory. He counts the number of incident pairs (v,e) in 2 different ways. Vertex v belongs to deg(v) pairs, where deg(v) (the degree of v) is the number of edges incident to it. Therefore the number of incident pairs is
the sum of the degrees. However, each edge in the graph belongs to exactly two incident pairs, one for each of its endpoints; therefore, the number of incident pairs is 2|E|.

This lemma is very important as it provides a theoretical framework for our project. Most of the explanations of our results are built upon the Handshaking Lemma.

3. Methodology

- To read up on relevant materials regarding graph theory and find theorems that can be used in our project

    - Handshaking Lemma (Degree Sum formula), which provides a theoretical framework for our project

- To research on related methods and find a generalised formula to solve the riddle
    - Discover a general formula to find the highest possible floor a control room can be on
    - Read up on different methods that people have used to solve the original problem and adopt them if possible to solve our research questions

- To experiment with new variables
    - Adding new control rooms
    - Changing the degree of control rooms
    - Changing the degree of non-control rooms

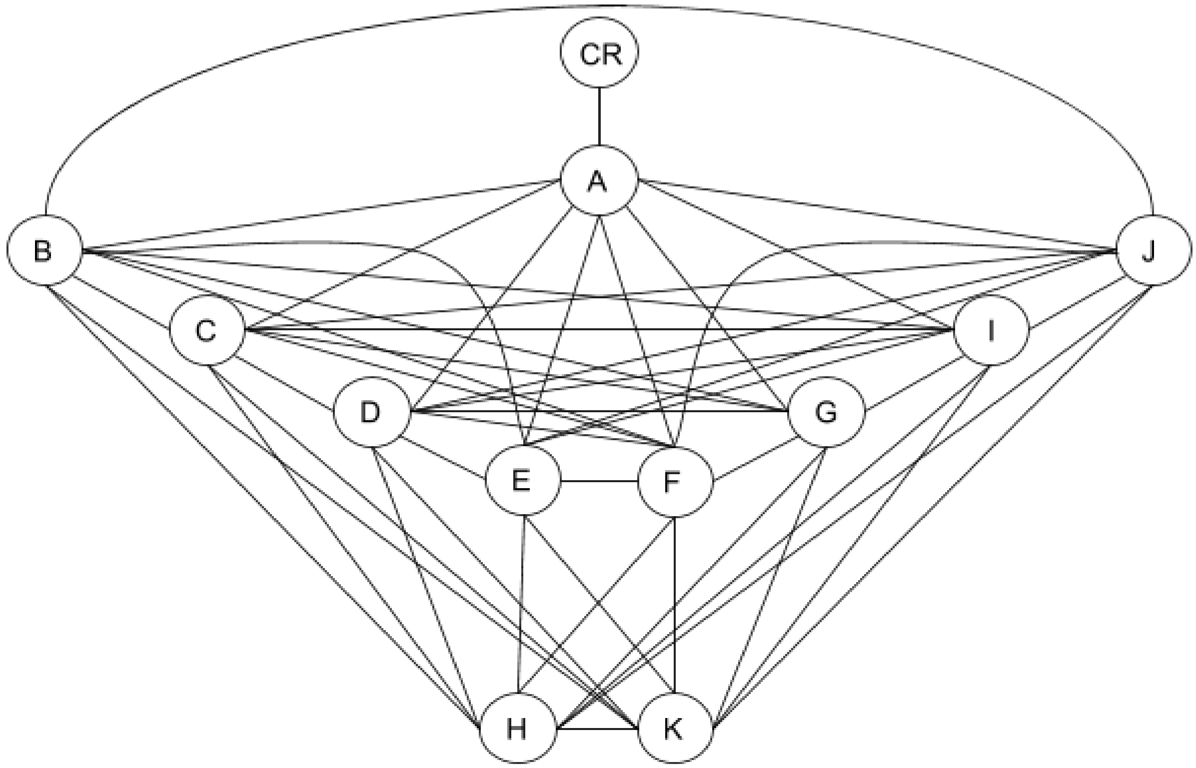
4. Results and Findings

Our general formula to calculate sum of degrees: $\boldsymbol{m}$ ($\boldsymbol{n}$ - $\boldsymbol{c}$) + $\boldsymbol{kc}$ = $\boldsymbol{p}$

4.1 Research Question 1

RQ 1: What is the highest possible floor the control room can be on if each room (except for the control room) is connected to $\boldsymbol{m}$ other rooms?

We approached this question by drawing the graph (Nodes representing rooms while edges represent the connections between the rooms) while varying the degree of each non-control room, $\boldsymbol{m}$, from $\boldsymbol{m}$ = 1 to $\boldsymbol{m}$ = 9.



(Fig 4.1 Graph of ***m*** = 9)

By following a series of steps, we are able to draw any graph of varying ***m***,

1. Connect control room to Room A
2. Connect Room A to (***m*** -1) rooms (Let the (***m*** - 1) rooms be set V)

3. Connect rooms in set V with each other
4. Draw 2 more vertices, representing 2 more rooms (Only apply when $m > 1$)
5. Connect rooms in set V to the 2 new vertices
6. Connect the 2 vertices together

***Note that this is a much more simplified way of drawing out the graph**

After drawing out the graphs, we tabulated our results as shown below (Fig 4.2)

Number of rooms each room is connected to, m	Highest floor that can satisfy conditions, n
1	2
3	6 = 3 + 3
5	8 = 5 + 3
7	10 = 7 + 3
9	12 = 9 + 3

Fig 4.2 Tabulated results of varying m

A relationship is observed between the degree of each non-control room and the highest possible floor a control room can be on. The highest possible floor the control room denoted by $m + 3$, where $m > 1$. It can be simplified to this:

$$n = \begin{cases} 2, & \text{If } m = 1 \\ m + 3, & \text{If } m > 1, \text{ where } m \text{ is odd} \\ \text{Undefined}, & \text{If } m \text{ is even} \end{cases}$$

We noticed that there are no graphs that are able to fulfill all the conditions when m = Even integer. The general formula is $m (n - c) + kc = p$. Since the degree of control room, k is 1, the no. of control room, c is 1, and since p is an even integer based on the Handshaking Lemma (2.2), the formula is

$m (n - 1) + 1$ = Even Integer.

When m = Even,

- $m (n - 1) + 1$ = Odd integer

When m = Odd,

- $m (n - 1) + 1$ = Even / Odd integer, depending on the nature of n
    - When n = Odd,
        - $m (n - 1) + 1$ = Odd integer
    - When n = Even,
        - $m (n - 1) + 1$ = Even integer

Applying the Handshaking Lemma (2.2), $\sum_{v \in V} deg(v) =$ Even integer. Hence, the only possibility to achieve an even sum of degree is when ***m*** = Odd and ***n*** = Even. Hence, this explains why the degree of a non-control room cannot be even and that the highest possible floor the control room can be on for ***m*** of varying values, will always be even.

When ***m*** = 1, ***n*** = 2. This does not follow the pattern of ***n*** = ***m*** + 3. The graph of ***m*** = 1 only involves the control room and a non-control room (Fig 4.3)

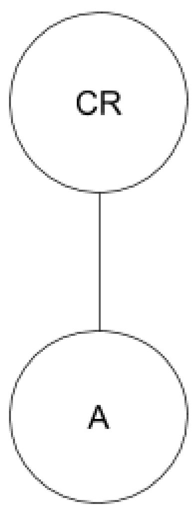


Fig 4.3 Graph of ***m*** = 1

When ***m*** = 1, the degree of the control room is the same as the degree of room A. Hence, there is no need for an additional room and ***n*** = 2, giving rise to this anomaly.

4.2 Research Question 2

RQ2: How will the solution change if each control room is connected to ***k*** other rooms?

Using the same approach as Research Question 1 (4.1), we drew out graphs of ***k*** = 1 to ***k*** = 6. Graph of ***k*** = 6 (Fig 4.4).

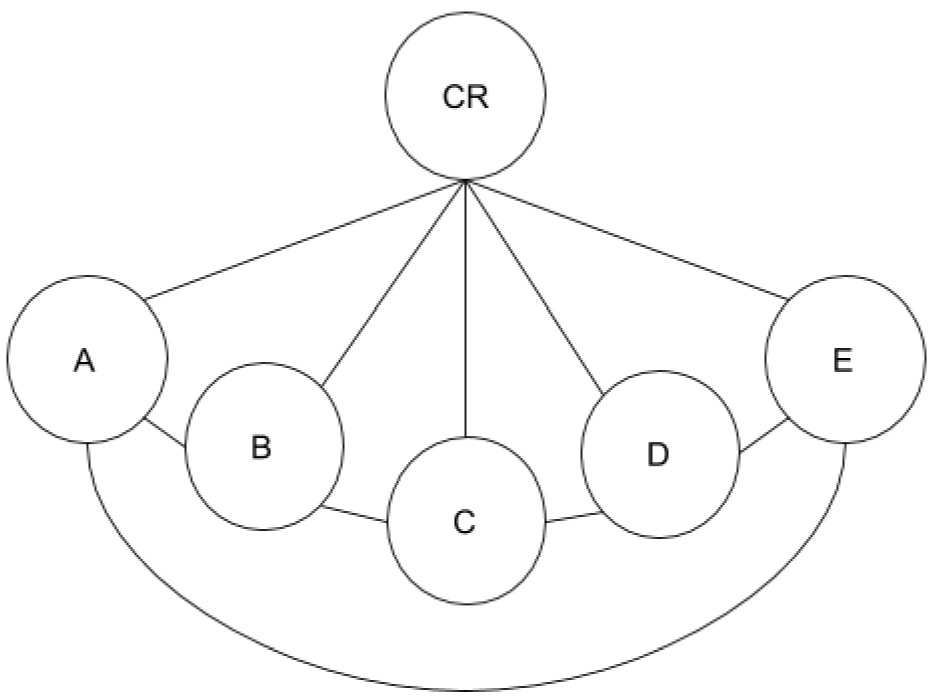


Fig 4.4 Graph of ***k*** = 6

By following a series of step, we are able to draw any graph of varying values of ***k***, when ***k*** > 2,

1. Connect CR to ***k*** rooms
2. Connect the non control rooms with each other

After drawing out the graphs, we tabulated our results as shown below (Fig 4.5).

Degree of control room, ***k***	Highest floor that can satisfy conditions, ***n***
1	6
2	5
3	4
4	5
5	6
6	7

Fig 4.5 Tabulated results of varying ***k***

After altering the degree of the control room, ***k,*** we noticed a relationship between the degree of the control room and the highest possible floor the control room can be on. This can be represented as $n = k + 1$. This is excluding cases $k = 1$ or $k = 2$. It can be simplified into this,

$$\boldsymbol{n} = \begin{cases} 6, & \text{If } \boldsymbol{k} = 1 \\ 5, & \text{If } \boldsymbol{k} = 2 \\ \boldsymbol{k} + 1, & \text{If } \boldsymbol{k} \geq 3 \end{cases}$$

We realised that the $\boldsymbol{n}$ = Even when $\boldsymbol{k}$ = Odd, while $\boldsymbol{n}$ = Odd when $\boldsymbol{k}$ = Even.

Using the same approach as RQ 1, $\sum_{v \in V} deg(v) = 3(\boldsymbol{n} - 1) + \boldsymbol{k}$.

- When $\boldsymbol{k}$ = Even,
  - $3(\boldsymbol{n} - 1) + \boldsymbol{k}$ = Odd / Even integer
- When $\boldsymbol{k}$ = Odd,
  - $3(\boldsymbol{n} - 1) + \boldsymbol{k}$ = Odd / Even integer

It can be seen that the sum of degree is dependent on both $\boldsymbol{n}$ and $\boldsymbol{k}$, hence,

- When $\boldsymbol{n}$ = Odd,
  - $3(\boldsymbol{n} - 1)$ = Even integer
  - $3(\boldsymbol{n} - 1) + \boldsymbol{k}$ = Even integer, $\boldsymbol{k}$ = Even
- When $\boldsymbol{n}$ = Even,

- ○ $3(n - 1)$ = Odd integer
- ○ $3(n - 1) + k$ = Even integer, k = Odd

Thus, to satisfy the Handshaking Lemma, it can be deduced that,

- n = Even when k = Odd
- n = Odd when k = Even

There are two anomalies $k = 1$ and $k = 2$. General pattern of graph of $k > 2$ shown below (Fig 4.6)

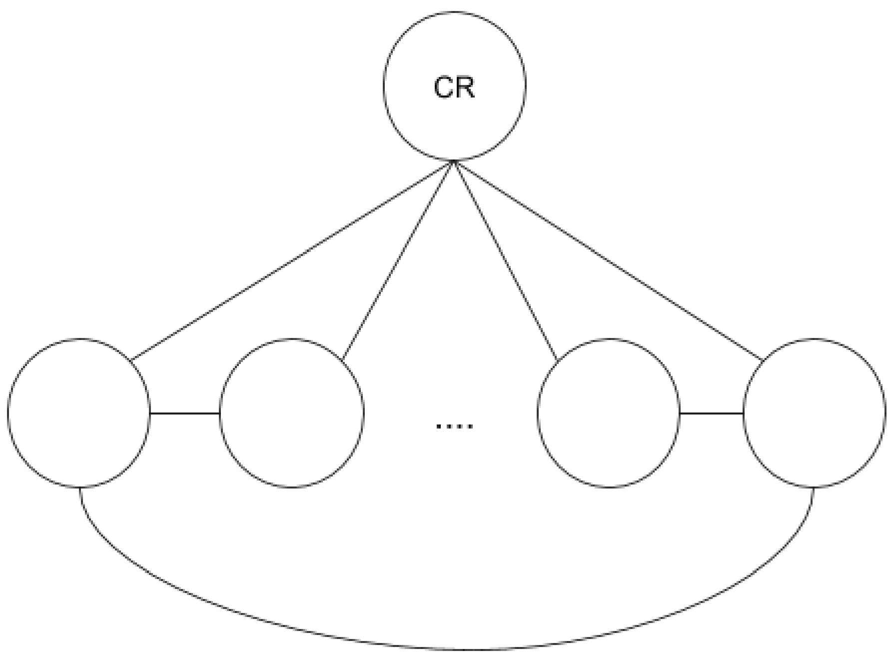


Fig 4.6 General pattern of graph with $k > 2$

The dots represent ($\boldsymbol{k}$ - 4) rooms which are connected to the CR. When $\boldsymbol{k}$ = 1 or 2, the general pattern does not apply. This is because $\boldsymbol{k} < \boldsymbol{m}$. When the control room is connected to $\boldsymbol{k}$ rooms when $\boldsymbol{k}$ = 1 or 2, the non-control rooms do not meet the condition of having a degree of 3, hence more non-control rooms have to be introduced to the graph, giving it a higher $\boldsymbol{n}$ value, not following the trend of $\boldsymbol{n} = \boldsymbol{k} + 1$. When $\boldsymbol{k} \geq 3$, $\boldsymbol{k} \geq \boldsymbol{m}$. The general pattern of the graph applies when $\boldsymbol{k} \geq \boldsymbol{m}$. Unlike when $\boldsymbol{k}$ = 1 or 2, when the CR is connected to $\boldsymbol{k}$ non-control rooms, the non-control rooms are able to form connections with each other, fulfilling the condition of $\boldsymbol{m} = 3$.

4.3 Research Question 3

How will the solution change if number of control rooms, $\boldsymbol{c}$, is changed?

Using the same approach as RQ 1 and RQ 2, we drew out graphs of $\boldsymbol{c}$ = 1 to $\boldsymbol{c}$ = 6. Graph of $\boldsymbol{c}$ = 6 show below (Fig 4.7).

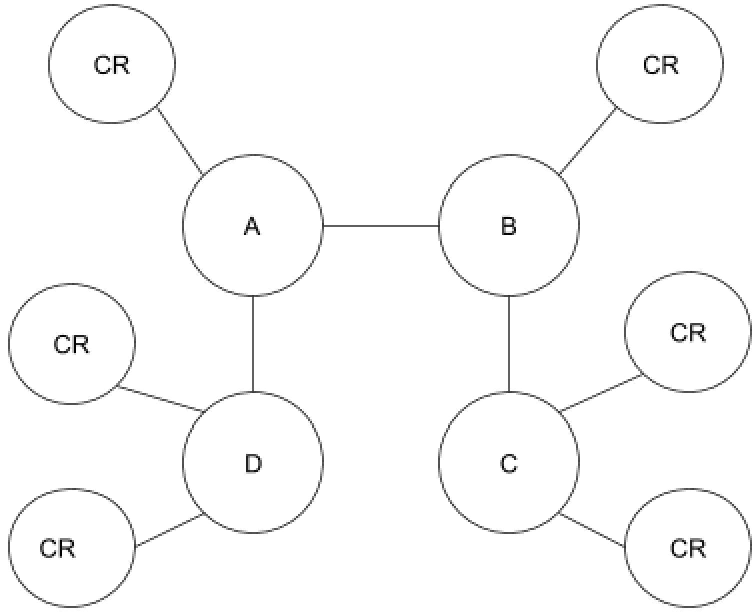


Fig 4.7 Graph of ***c*** = 6

We tabulated our results as shown below (Fig 4.8)

Number of control room, ***c***	Highest floor that can satisfy conditions, ***n***
1	6
2	6
3	4 = 2(3 - 1)
4	6 = 2(4 - 1)
5	8 = 2(5 - 1)
6	10 = 2(6 - 1)
7	12 = 2(7 -1)

After working out the graph for different numbers of control room, ***c*** , we noticed a relationship between ***c*** and ***n***. This can be represented by ***n = 2c*** - 2. This is excluding cases ***c*** = 1 or ***c*** = 2. It can be simplified into this,

$$
\boldsymbol{n} = \begin{cases} 6, & \text{If } \boldsymbol{c} = 1 \text{ or } 2 \\ 2\boldsymbol{c} - 2, & \text{If } \boldsymbol{c} > 2 \end{cases}
$$

In order for ***p***, the sum of degree, to be even, there can be 2 cases

- 3 (***n*** - ***c***) + ***c*** = Even integer
  - 3 (***n*** - ***c***) = Even integer, ***c*** = Even integer
  - 3 (***n*** - ***c***) = Odd integer, ***c*** = Odd integer

Case 1: If 3 (***n* - *c***) **=** Even, and ***c*** = Even,

- (***n*** - ***c***) = Even integer
- Since ***c*** is even in Case 1, ***n*** is even

Case 2: If 3 ***(n* - *c)*** = Odd, and ***c*** = Odd

- (***n* - *c***) = Odd integer
- Since ***c*** is odd in Case 2, ***n*** is even

Hence, we can conclude that since $\sum_{v \in V} deg(v)$ = Even, ***n*** must be even regardless of the value of ***c***. This explains why the highest possible floor the control room can be on will always be even.

We also noticed a general pattern of how the graph will look like when we vary the value of ***c***.

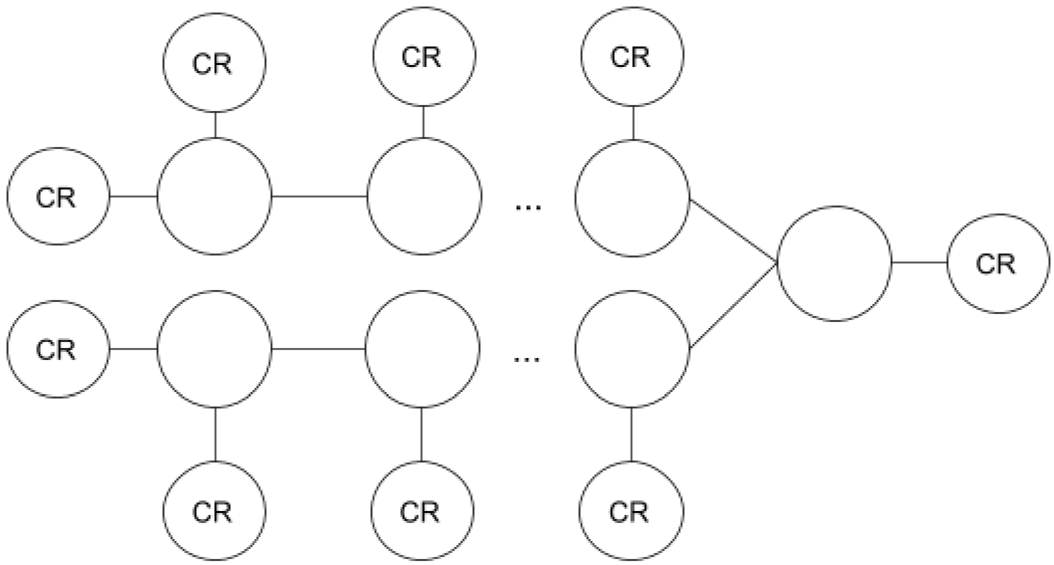


Fig 4.8 Graph of ***c*** = Odd, ***c*** > 2

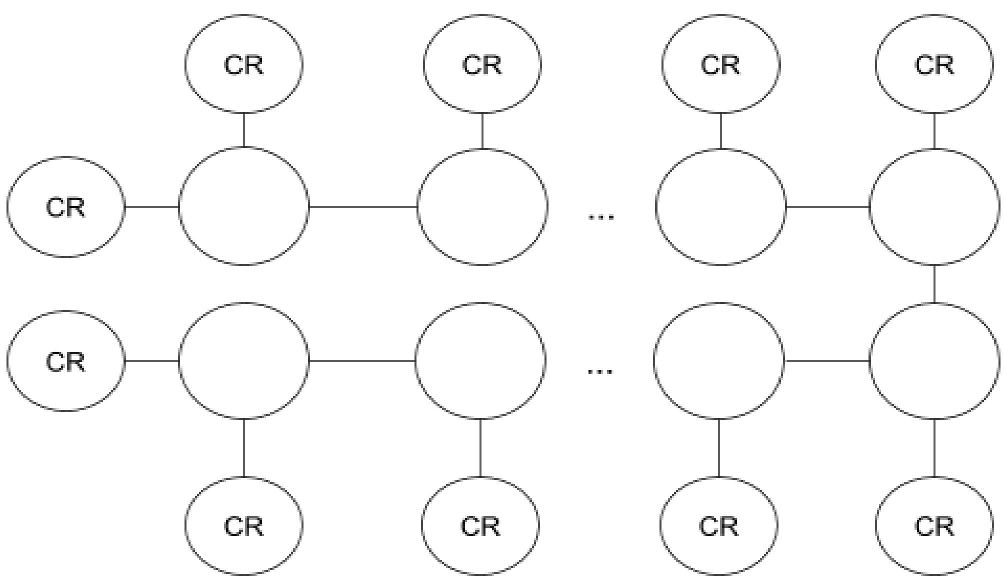


Fig 4.9 Graph of ***c*** = Even, ***c > 2***

After rearranging the graphs, we found 2 different but similar general patterns for graphs when c = Odd (Fig 4.8) and c = Even (Fig 4.9). There is always a pair of non-control rooms which is connected to 2 control rooms each. The no. of non-control rooms can be expressed as $c - 2$ since there is a recurring pattern of 1 non-control room connected to 1 control room. Hence, n is represented as $2c - 2$ since the no. of control rooms is c and the no. of non-control rooms is $c - 2$ and $c + (c - 2) = 2c - 2$.

The value of n when $c = 1$ and 2 does not follow the trend. This is because there is an insufficient of rooms for non-control rooms to connect to and fulfill the condition of $m = 3$. Hence, more non-control rooms have to be introduced, giving to a n value that does not follow the trend.

5. Conclusion

From Research Question 1,

We inferred that $n = m + 3$, when $m > 1$ and m = Odd. When m is an even integer, n is undefined as it does not following the Handshaking Lemma.

Anomalies: $m = 1$, $n = 2$

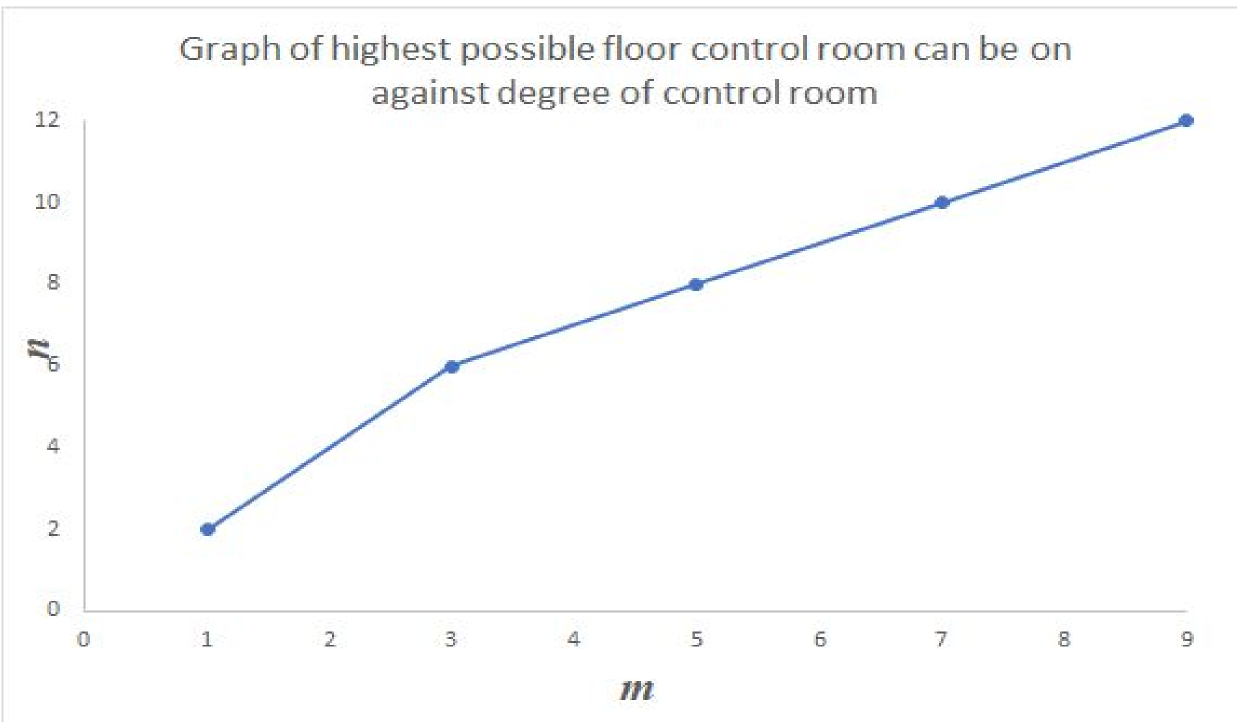


Fig 5.1: Graph of n against m

From Research Question 2, we inferred that **n** = ***k*** + 1, when ***k*** > 2.

Anomalies: ***k*** = 1, **n** = 6 and ***k*** = 2, ***n*** = 5

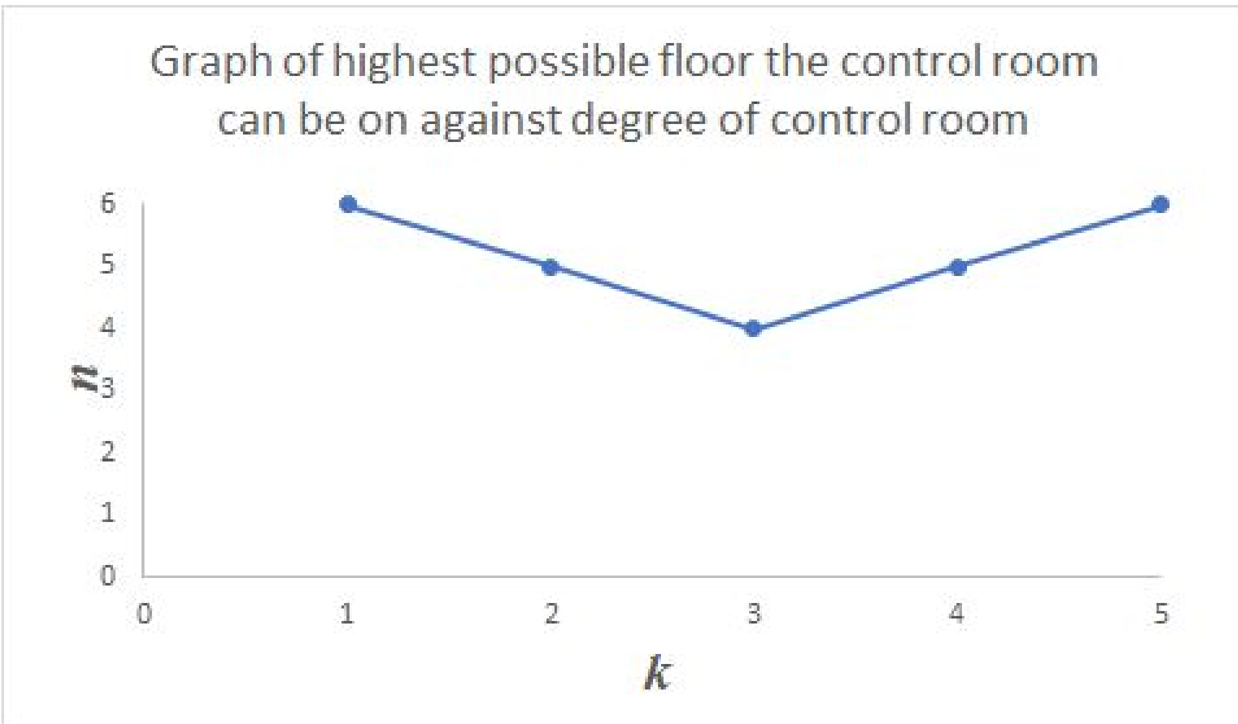


Fig 5.2: Graph of ***n*** against ***k***

From Research Question 3, we inferred that $n = 2c - 2$, when $c > 2$.

Anomalies: $c = 1$, $n = 6$ and $c = 2$, $n = 6$

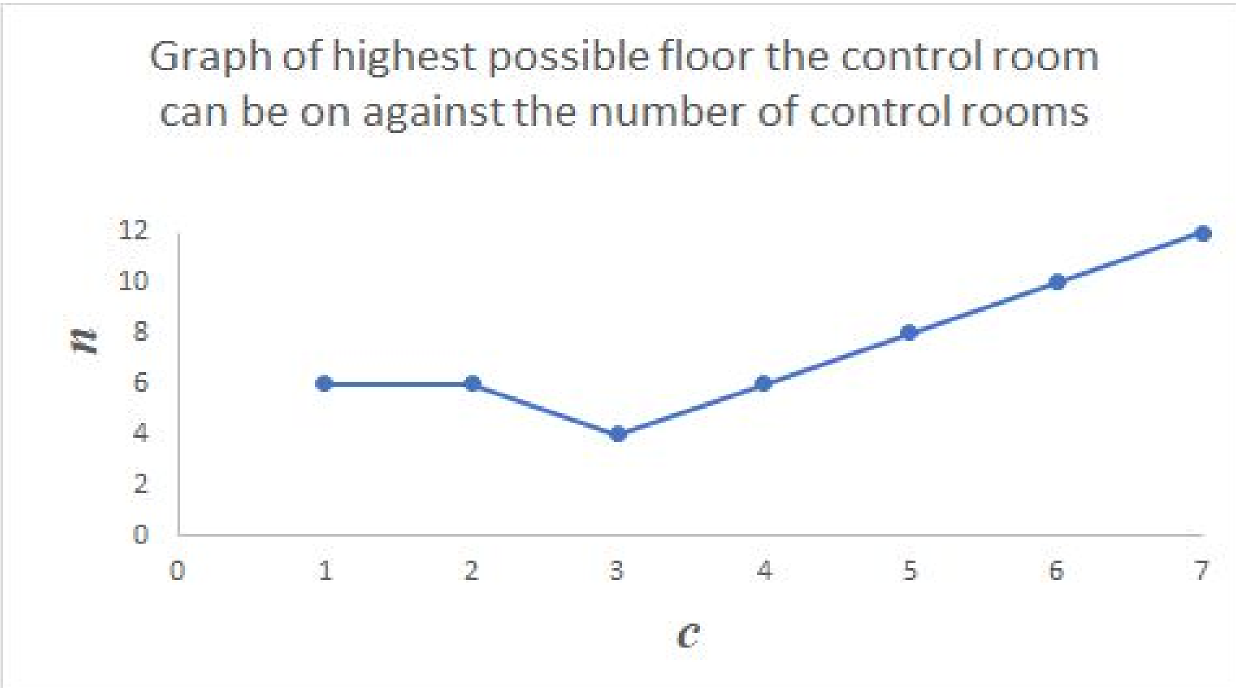


Fig 5.3: Graph of ***n*** against ***c***

5.1 Implications and Significance

After plotting graphs for the research problems (Fig 5.1, 5.2 and 5.3), we realise that aside for the exceptions in the beginning, the graph starts to become a straight line. We were able to infer that all three variables ***m***, ***k***, ***c*** are linearly related with ***n***. Since it is a linear trend, there is a constant increase in the highest possible floor the control room can be on. When value of ***k*** and ***c*** increases by 1, ***n*** increases by 1 and 2 respectively. When value of ***m*** increases by 2, ***n*** increases by 2 as well.

5.2 Limitations

- Did not take into consideration of the possibility of pseudo graphs and multigraphs
- Only changed one variable at a time
- Rooms are connected within the same floor

5.3 Project Extensions

- Consider the possibility of pseudo graphs and multigraphs
- Change two or more variables at a time and find out the general trend for it
- Instead of having only two types of rooms, introduce more types of rooms that have different conditions that we have to comply with
- Find the general pattern of the sum of degree of each graph for each variable changed

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