Polya's Orchard Problem

Group leader: Phua Jun Kai (2A227)

Group members: Ryan Lim Hong Wei (2A217)

Lee Chang Feng (2O110)

Nigel Wang Wen Le (2P333)

Hwa Chong Institution (High School)

1 Introduction

Polya's Orchard Problem is a subset of orchard problems in mathematics. Specifically, it investigates how thick the trunks of trees be in a regularly spaced orchard must be in order to completely block the view from the center and many interesting ideas are derived from this problem. This project aims to uncover the mathematical theories behind it and explore some variations.

This project was chosen due to the plethora of questions that can be asked relating to the problem and challenging concepts will be required for them.

1.1 Objectives

1) To understand the problem and its solutions mathematically
2) To investigate the effect of parameters in the problem
3) To investigate the variations of the problem

1.2 Research Questions

1) What is the solution to the original problem?
2) What is the probability that one's vision from the origin is blocked if he looks in a random direction?
3) Can we generalize the results for trees planted in a square orchard?

2 Literature Review

George Pólya first solved the problem the problem based on Andreas Speiser's method (Alexandru, 2006) while Ross Honsberger solved it using Minkowski's Theorem (Kruskal, 2005). They use integers and bound the given coordinates for the lattice points to check whether there are still lines of sight or whether the person standing at the origin has his vision completely blocked.

As it has only been used for a circular orchard, other scenarios can be investigated. Thomas Tracy Allen generalised the result to orchards with a non-integer radius, something the previous two solutions failed to address (Allen, 1986).

3 Study and Methodology

The intended approaches are doing comprehensive research to find more articles related to the problem, understanding the explanations by getting inspiration from the available methods and solving variations by applying research concepts. A light of sight is a ray from the origin that does not intersect any trees.

3.1 Results

Let the trees be circles of radius r and the radius of the entire orchard be R, which has to be a natural number. Trees are positioned on lattice points bounded by the circular orchard, including boundary points but excluding the tree at the origin.

3.1.1 Bounds of r for circular orchard

The bounds of r will first be specified. For the first case, where $R = 1$, the extreme case where the trees are touching each other diagonally is considered. By Pythagoras' Theorem, $2r = \sqrt{2}$ (Figure 1) and thus:

$$0 < r \leq \frac{\sqrt{2}}{2}, R: R = 1.$$

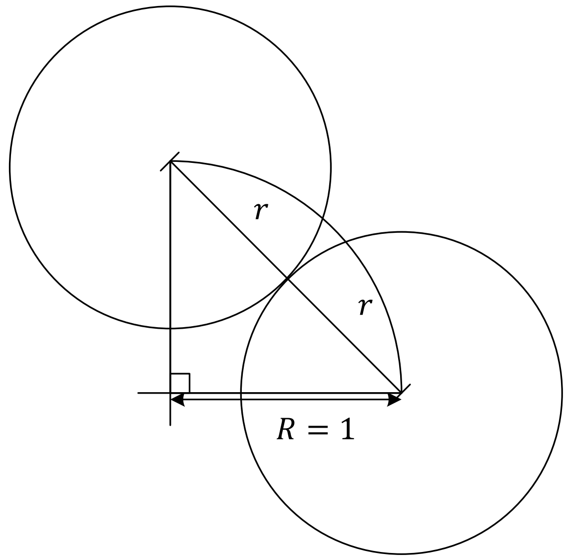

Figure 1

If $R > 1$, the horizontal and vertical distances are considered. Since the trees are positioned on lattice points, $2r = 1$ (Figure 2). Hence:

$$0 < r \leq \tfrac{1}{2}, R: R > 1, \;\; R \in \mathbb{N}.$$

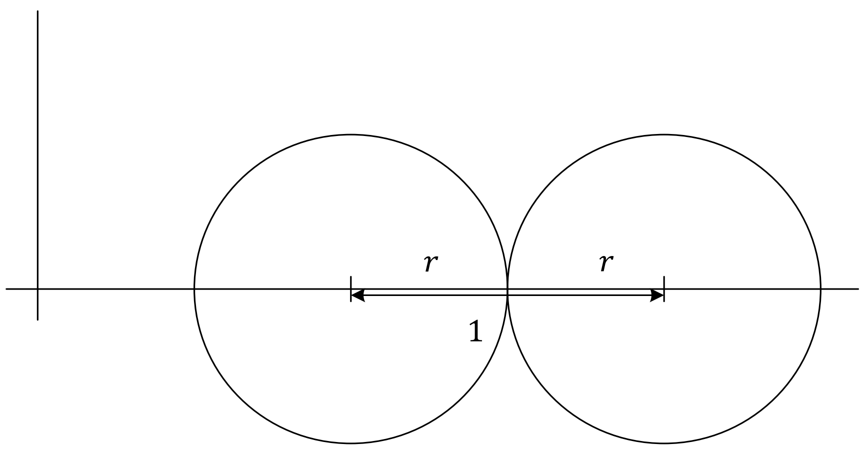

Figure 2

3.1.2 Proof that lines of sight exist in circular orchard if $r < \frac{1}{\sqrt{R^2+1}}$

This proof considers the trees at $(1,0)$ and $(R-1,1)$ that may block the line of sight from the origin to $(R,1)$, a point outside the orchard, as they are the closest trees (Figure 3).

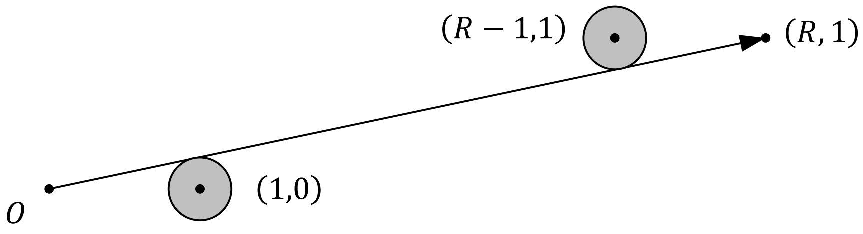

Figure 3

The points $(1,0)$, $(R,1)$, and the origin form a triangle (Figure 4). Let the value of r such that the trees are tangent to the ray be δ. The ray distance is $\sqrt{R^2+1}$ by Pythagoras' Theorem.

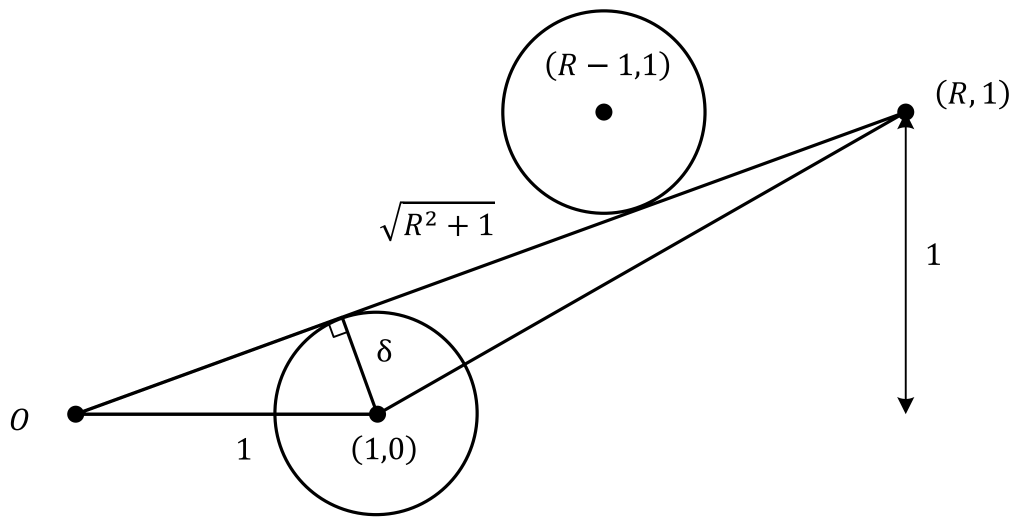

Figure 4

By calculating the area of the triangle using different bases and heights, the following equation is obtained:

$$\frac{\delta}{2}\sqrt{R^2+1} = \frac{1}{2}.$$

Hence,

$$\delta = \frac{1}{\sqrt{R^2+1}}.$$

For a line of sight from the origin to exit the orchard, r cannot be greater than δ. Therefore, there exists lines of sight if:

$$r < \frac{1}{\sqrt{R^2+1}}.$$

3.1.3 Proof that no lines of sight exist in circular orchard if $r > \frac{1}{R}$

Firstly, consider AB as an arbitrary diameter of the orchard. It is also a potential line of sight. Let $r = \frac{1}{R} + \varepsilon$, where $0 < \varepsilon < \frac{1}{2}$ as $0 < r \leq \frac{1}{2}$, and the length $AC = AF = DB = BE = \frac{1}{R} + \frac{\varepsilon}{2}$. As $r > AF$, any tree in the rectangle $CDEF$ will block AB. Minkowski's Theorem is used to prove that there will be trees in rectangle $CDEF$. It states that any convex region symmetric about its origin, with an area greater than 4, contains lattice points other than the origin itself. The area of the rectangle, $(AB)(CF)$ is (Figure 5):

$$(2R)2\left(\frac{1}{R} + \frac{\varepsilon}{2}\right) = 4 + 2R\varepsilon.$$

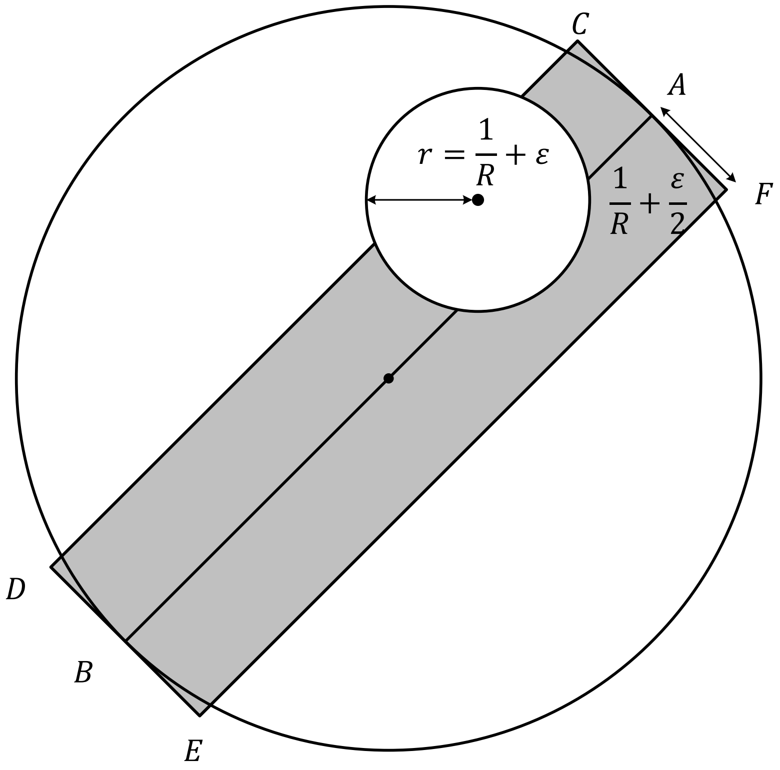

Figure 5

$4 + 2R\varepsilon$ is greater than 4 since ε is a positive number. The rectangle is a convex region that is symmetric about the origin and has and area greater than 4, so by Minkowski's Theorem, it contains lattice points other than at the origin. Therefore, there will be trees that will block the line of sight.

However, there is a technicality. The lattice points could be in the tiny sliver outside the orchard, which will not block the lines of sight as there are no trees there. $0 < \varepsilon < \frac{1}{2}$, $R > 1 \Rightarrow \frac{1}{R} + \frac{\varepsilon}{2} < 1$. Hence,

$$R^2 + \left(\frac{1}{R} + \frac{\varepsilon}{2}\right)^2 < R^2 + 1.$$

The coordinates of such a point must satisfy the following equation due to the distances, i.e.

$$R^2 < x^2 + y^2 < R^2 + 1$$

No integers, x, y can satisfy this inequality. $r = \frac{1}{R} + \varepsilon$ implies that there will be no lines of sight if:

$$r > \frac{1}{R}.$$

3.1.4 Probability that vision from the origin is blocked

The angle formed by the two lines of sight tangent to a specific tree is first found. By Pythagoras' Theorem, $OC = \sqrt{x^2 + y^2}$. By the Tangent-Radius Theorem, $\angle OAC = \frac{\pi}{2}\ rad$ (Figure 6). Hence, $\sin\frac{\alpha}{2} = \frac{r}{\sqrt{x^2+y^2}}$,

$$\therefore \alpha = 2\sin^{-1}\frac{r}{\sqrt{x^2 + y^2}}.$$

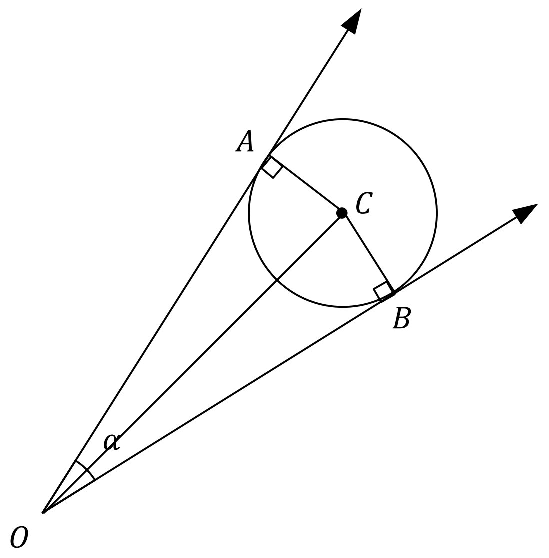

Figure 6

If $R > 3$, two cases for the bounds of r must be considered (Figure 7).

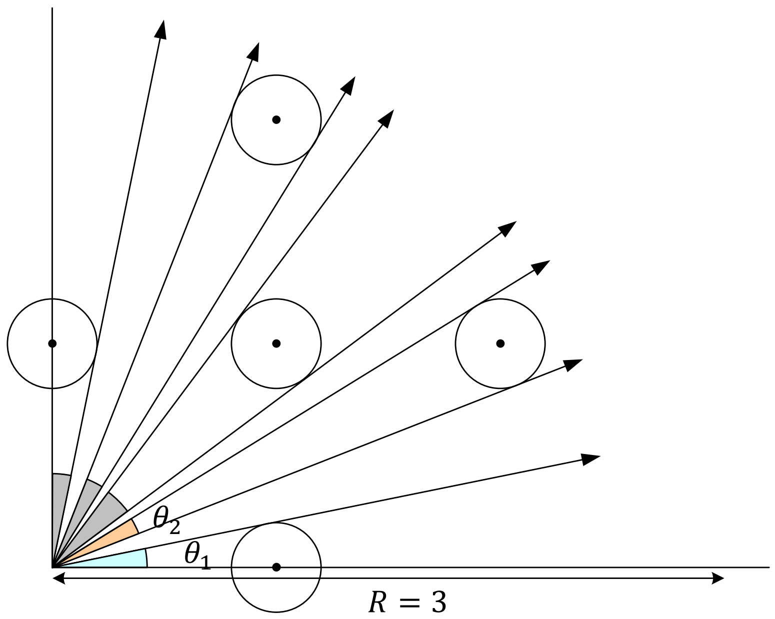

Figure 7

For θ_1, the formula $0 < r < \frac{1}{\sqrt{R^2+1}}$ is used. For θ_2, Pick's Theorem is used to find the area of a parallelogram for two such trees, i.e. $\frac{4}{2} + 0 - 1 = 1$. Using the area of a parallelogram and Pick's Theorem gives (Figure 8):

$$1 = 2\left(\frac{r}{2}\sqrt{(a+c)^2 + (b+d)^2}\right),$$

$$\therefore r < \frac{1}{\sqrt{(a+c)^2 + (b+d)^2}}.$$

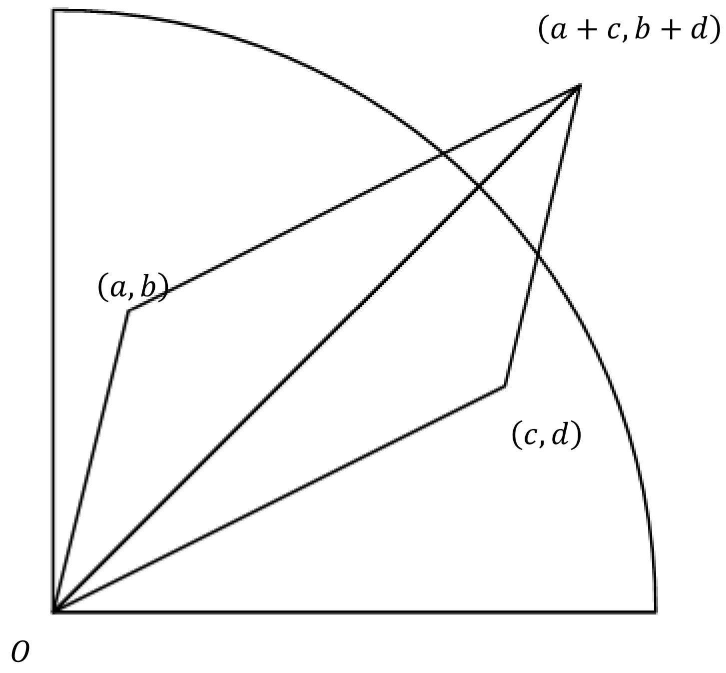

Figure 8

Let θ be the sum of all angles created by two tangent lines to a specific tree (Figure 9). For $R = 1$, using the previous finding,

$$\theta = 2\sin^{-1} r.$$

The condition for the tree's radius is $0 < \mathrm{r} < \frac{1}{\sqrt{1^2+1}} = \frac{1}{\sqrt{2}}$ for there to be a line of sight. Therefore, the probability of vision being blocked if the orchard radius is 1 is:

$$P_1 = \frac{4}{\pi}\sin^{-1} r.$$

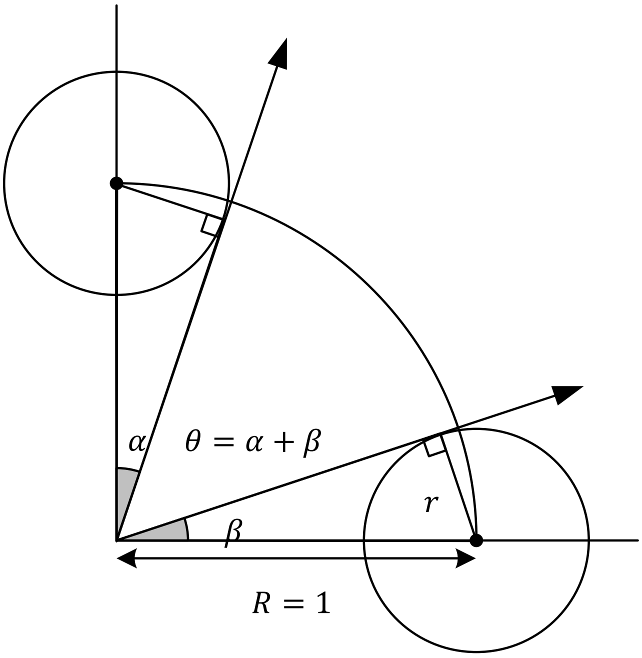

Figure 9

Using the same method for more cases generates Table 1.

R value	P_R
$R = 1$	$\frac{4}{\pi} sin^{-1} r$
$R = 2$	$\frac{4}{\pi}\left(sin^{-1} r + sin^{-1}\frac{r}{\sqrt{2}}\right)$
$R = 3$	$\frac{4}{\pi}\left(sin^{-1} r + sin^{-1}\frac{r}{\sqrt{2}} + 2\, sin^{-1}\frac{r}{\sqrt{5}}\right)$
$R = 4$	$\frac{4}{\pi}\left(\sin^{-1} r + \sin^{-1}\frac{r}{\sqrt{2}} + 2\sin^{-1}\frac{r}{\sqrt{5}} + 2\sin^{-1}\frac{r}{\sqrt{10}} + 2\sin^{-1}\frac{r}{\sqrt{13}}\right)$
$R = n$	...

Table 1

The terms in the table are dependent on the number of trees on irreducible points meaning they are visible trees. Let (a, b) be an irreducible point. The Stern-Brocot tree generates all such points. The values in the radical signs correspond to $a^2 + b^2$.

3.1.5 Bounds of r for square orchard

For this solution, a square orchard is considered (Figure 10). Lines of sight exist if $r < \frac{1}{\sqrt{(R+1)^2+1}}$ and no lines of sight exist if $r > \frac{1}{R}$. The bound of r is stated below:

$$0 < r \leq \frac{1}{2} \;\; for\ \forall R.$$

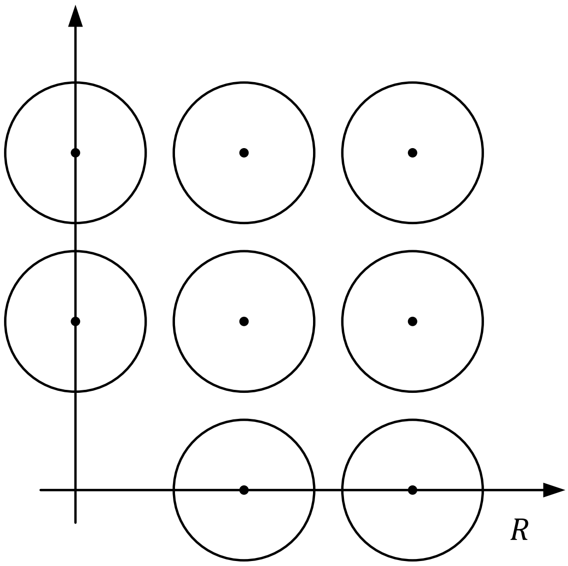

Figure 10

3.1.6 Proof that lines of sight exist in square orchard if $r < \frac{1}{\sqrt{(R+1)^2+1}}$

To proof that $r < \frac{1}{\sqrt{(R+1)^2+1}}$ has a line of sight, the trees at $(R, 1)$ and $(1,0)$ that may block the line of sight from the origin to $(R + 1,1)$ are considered as the closest trees. Let the value of r such that the trees are tangent to the ray be δ. By Pythagoras' Theorem, the line of sight is $\sqrt{(R+1)^2+1}$. The area of triangle is $\frac{\delta}{2}\sqrt{(R+1)^2+1}$ which is also equal to $\frac{1}{2}$. Therefore, lines of sight exist if:

$$r < \frac{1}{\sqrt{(R+1)^2+1}}.$$

3.1.7 Proof that no lines of sight exist in square orchard if $r > \frac{1}{R}$

Let θ be the angle produced by the rectangle in the square orchard (Figure 11). $0 \le \theta \le \frac{\pi}{4}$ due to symmetry. $OA = Rsec\theta$ and thus the length of the rectangle will be $2Rsec\theta$. $AF = \frac{1}{Rsec\theta} + \frac{\varepsilon}{2}$, and $r = \frac{1}{Rsec\theta} + \varepsilon$. The area of the rectangle is:

$$(2Rsec\theta)2\left(\frac{1}{Rsec\theta} + \frac{\varepsilon}{2}\right) = 4 + 2R\varepsilon sec\theta > 4.$$

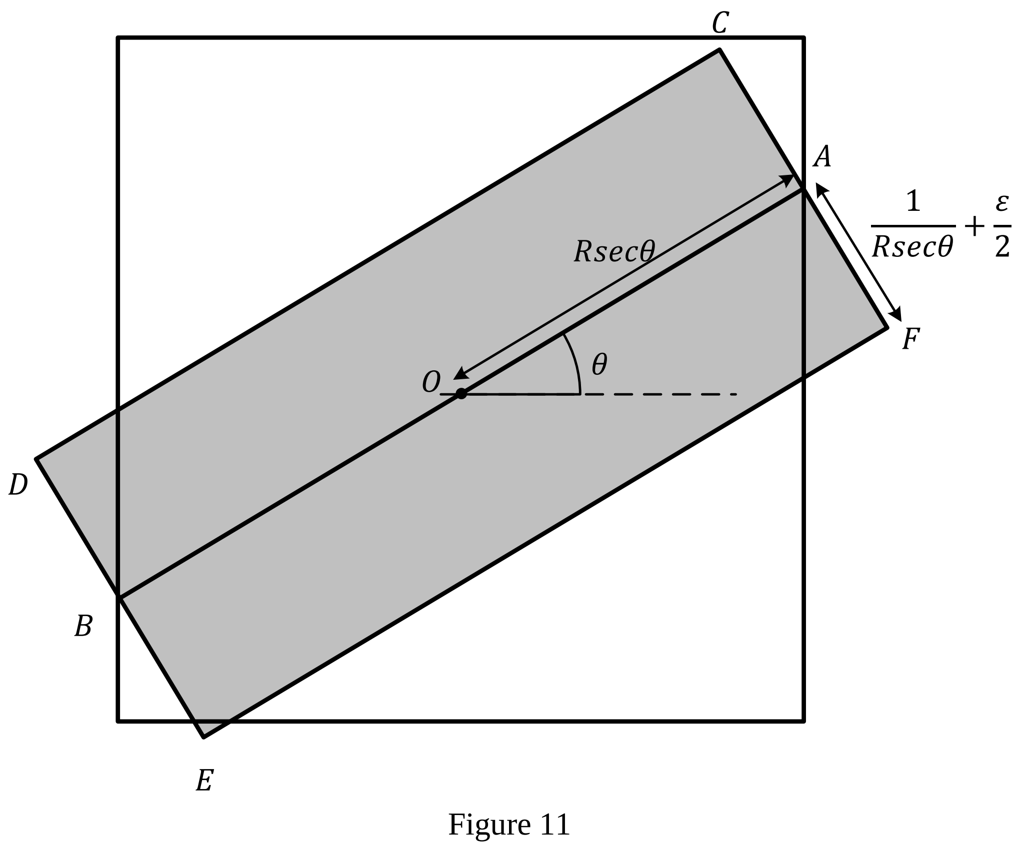

Figure 11

By Minkowski's Theorem, there are trees that will block the lines of sight. Since θ is increasing from 0 to $\frac{\pi}{4}$ rad, $\frac{1}{Rsec\theta}$ decreases as θ increases. $r = \frac{1}{Rsec\theta} + \varepsilon$ gives the inequality $r > \frac{1}{Rsec\theta}$. The largest value of $\frac{1}{Rsec\theta}$ is considered to ensure that there is no line of sight. The extreme values of θ are shown:

$$\frac{1}{Rsec\left(\frac{\pi}{4}\right)} = \frac{1}{R\sqrt{2}},$$

$$\frac{1}{Rsec(0)} = \frac{1}{R}.$$

$\frac{1}{R}$ is able to ensure that there exists no line of sight for all θ values. Therefore, the lines of sight do not exist if:

$$r > \frac{1}{R}.$$

4 Conclusion

In conclusion, Ross Honsberger's method was used to solve the original problem. There are different probabilities that one's sight from the origin will be blocked if he looks in a random direction for different values of the orchard's radius and noticeable pattern can be observed as the orchard radius changes. This project can be further extended by considering a three-dimensional orchard and different lattices, then comparing them to the original solution. A tighter bound could also be achieved for the square orchard.

5 References

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