KILLER SUDOKUS

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2A3

Contents

1. Introduction

Sudoku is a common puzzle that almost everyone knows of. But have you heard of killer sudokus? Killer sudoku is a puzzle that combines elements of sudoku. The sum of the numbers in the cells of a cage are indicated at a corner.

1.1 Terminology

Terms	Definition
cell	a single square that contains one number in the grid
row	a horizontal line of 9 cells
column	a vertical line of 9 cells
cage	the grouping of cells denoted by dotted lines or individual colours
nonet	a 3×3 grid of cells, outlined by bolder lines
house	any row, column or nonet which contains 9 cells and exactly 1 of each number from 1-9

1.2 Research Questions

1. How many ways are there to arrange trimino cages in 9×9 killer sudokus, given that the cages cannot cross the nonet?

2. How many ways are there to arrange trimino cages in killer sudokus, with at least 1 cage each containing multiplication, subtraction and addition for 9x9 killer sudokus, given that the cages cannot cross the nonet?

3. How many ways are there to arrange using only triominoes and 1x1 cages in 9x9 killer sudokus, given that the cages cannot cross the nonet?

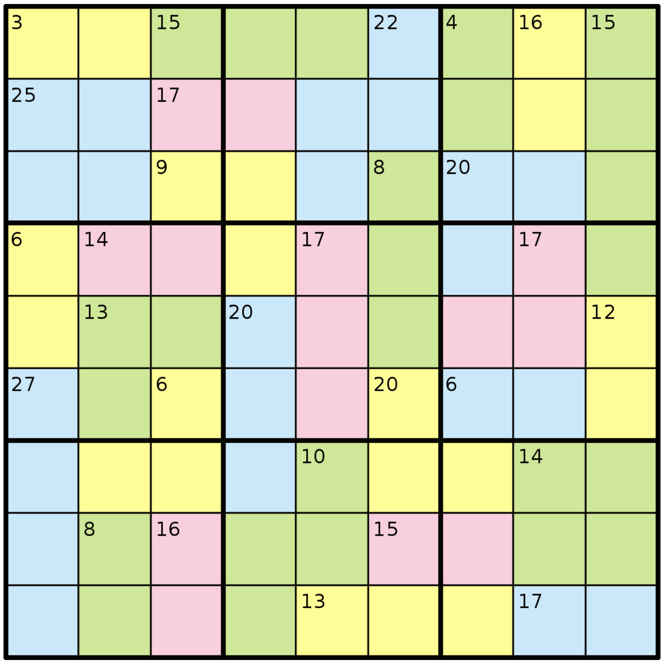

Figure 1

This is what a killer sudoku looks like.

1.3 Rules

The objective is to fill the grid with numbers from 1 - 9. However, certain conditions must be met.

- each row, column, and nonet contains each number exactly once
- the sum of all numbers in a cage must match the small number in its corner

2 Literature Review

1.2.1. Thomas Petty

"Final Report - "Killer Sudoku Solver"

Retrieved from:

https://www.cs.cf.ac.uk/PATS2/@archive_file?c=&p=file&p=558&n=final&f=1-1119707_Final_Report.pdf

It is about a report on creating an application that solves killer sudokus by implementing a number of different solving strategies in his solution, and also performing a series of experiments using different combinations or orderings of strategies to determine which combinations are the most effective or run the fastest. This website contains some methods of solving killer sudokus.

1.2.2. Weisstein, Eric W.

“Domino Tiling”

Retrieved from: http://mathworld.wolfram.com/DominoTiling.html

This is quite similar to research question 1, except that the method it used was to create an algebraic equation to find the possible ways to tile 1x2 dominoes (i.e. cages) into 2n x 2n squares, so it does not work for 9x9.

3 Methodology

For each research question, we first experimented with the 4x4, then moved on to the 9x9. We attempt to come up with a way to simplify the questions first like focusing on the 3x3 cage within the 9x9. If we are unable to find a method we would resort to brute force.

4 Results

4.1 Ways to arrange cages

4.1.1 On a 4x4 Grid

We will be starting off with 4x4 first. To recap, this is the question:

How many ways are there to arrange 1x2 cages in 4x4 killer sudokus?

Notice that the sum of numbers in the cages are redundant, hence we can simply remove them to make it easier. Firstly, we split the 4x4 into four 2x2 squares, as seen in Figure 1 below. Note that the cross coloured green is the border of the 2x2 squares.

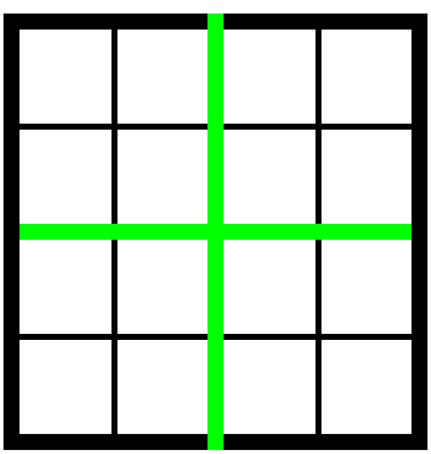

Figure 2.1

We now have two cases:

1. The 1x2 tiles does not cross the border of the 4 2x2s
2. The 1x2 tiles cross the border of the 4 2x2s

4.1.1.1 Case 1

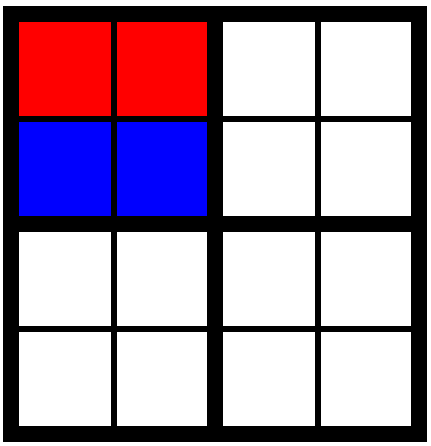

Figure 3.1

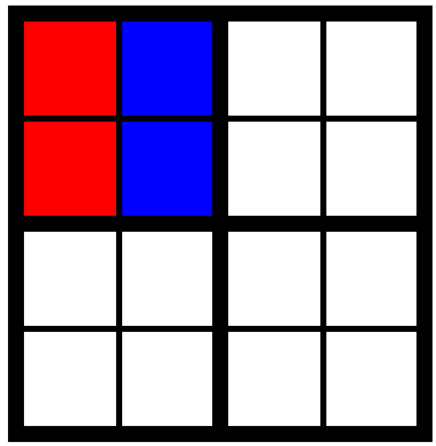

Figure 3.2

Firstly we start with case 1,it is obvious that there are only 2 possible arrangements for each 2x2, as shown by figure 3.1 and figure 3.2.As there are 4 2x2s, the total number of configurations for case 1 is 2x2x2x2, which is 16.

4.1.1.2 Case 2

Now, what do we do now for case *2*? We know that it is not possible that an odd number of 1x2s crosses the border of each 2x2 square.

Why is that so? Well, it will leave the 2x2 with an odd number of cells, either 3 or 1, meaning that 1x2 cages would be unable to fill the area. For example, in figure 3.3 (page 6), if there is a 1x2 cage crossing the border, then it will be such that it is impossible to cover the 2x2 with 1x2 cages, and will leave a cell empty, which is unacceptable.

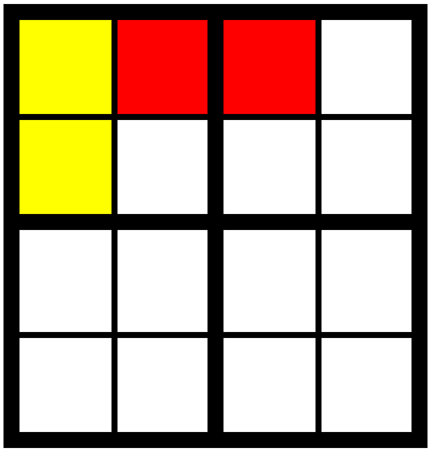

Figure 3.3

Therefore, there must be an even number of 1x2 cages crossing the border.

With this conclusion, we can once again split this into a few cases:

a) There are **2** 1x2 cages crossing the border

b) There are **4** 1x2 cages crossing the border

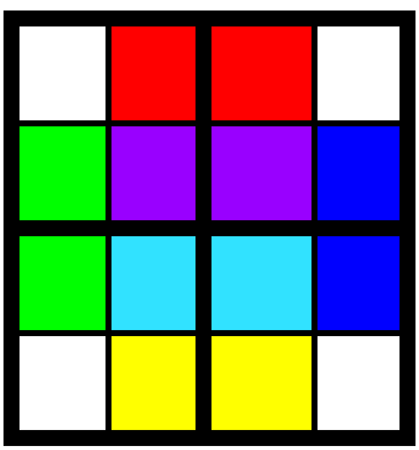

Figure 3.4

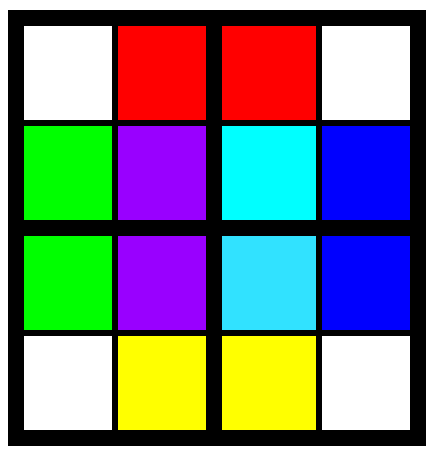

Figure 3.5

We cannot have 6 1x2 cages crossing the border as shown in Figure 3.4 and Figure 3.5

4.1.1.2.1 Case 2a

Factoring in rotation, we realised that there were many ways for the 2 1x2 cages crossing the border to be arranged. However, from our previous conclusion that there must be an even number of 1x2 crossing the border for each 2x2 square *(see page 5)*, we can see that the 2 1x2 cages must connect to both squares*(i.e. figure 4.1)*

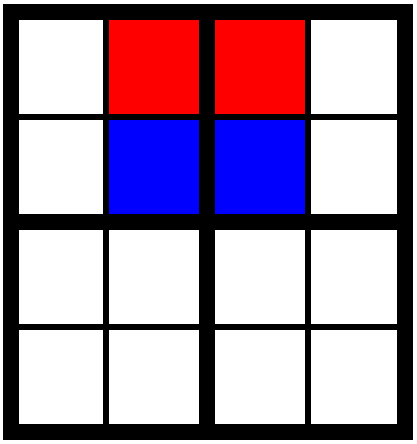

Figure 4.1

Hence, there must be 1x2 cages, represented by the yellow squares*(figure 4.2)* at that area. That leaves us with 2 2x2 boxes, and there are 4 ways to arrange 1x2 cages inside, without crossing the border. With rotation, there will be 4x4=16 permutations.

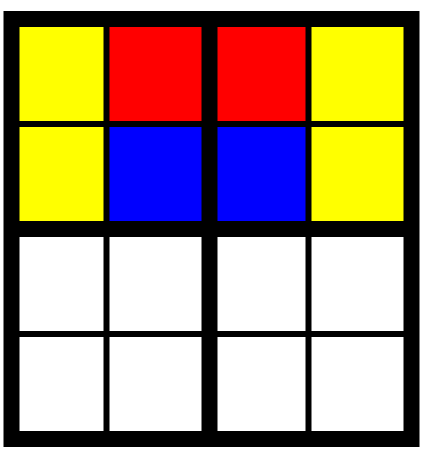

Figure 4.2

4.1.1.2.2 Case 2b

We realised that there needed to be a 1x2 cage at the edge, as only 2 1x2 cages can fit at the centre. Let's say the 1x2 cage is the at the yellow area*(figure 5)*.

Hence, there cannot be 1x2 cages at the red area, and the only possibility is the blue area. There are 2 possible arrangements for it. However, we need to account for rotation, and since it is symmetrical, we only need to multiply 2x2=4 to get the answer.

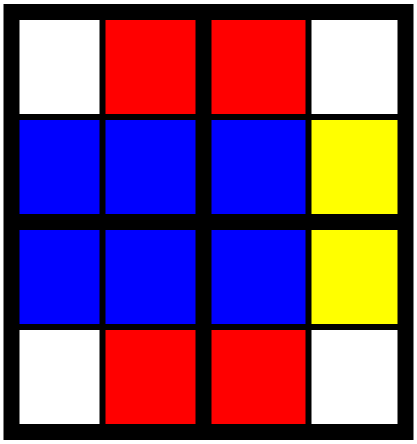

Figure 5

Talking all the cases, the total number of possible arrangements for 1x2 cages in 4x4 killer sudokus is 16+16+4=36.

4.1.2 On a 9x9 Grid

For the 9x9 killer sudoku, we will use L-shaped cages*(figure 6.1)* and 1x3 cages*(figure 6.2)*, such that they will not cross the borders of a 3x3 box.

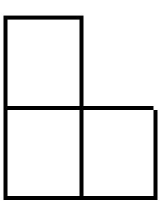

Figure 6.1

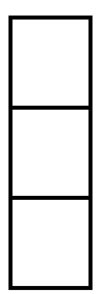

Figure 6,2

We start by focusing on 1 3x3, and consider the possible cases of how we can place 1x3 cages and L-shaped cages into the 3x3 box.

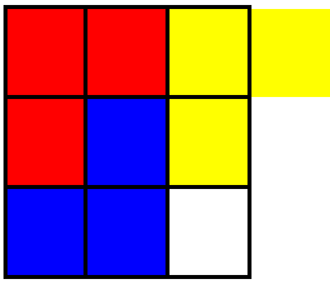
Figure 7.1

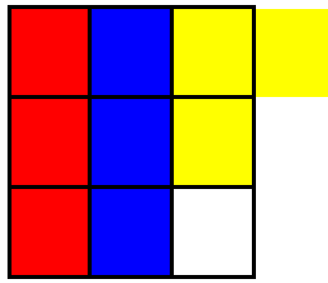
Figure 7.2

We know that we cannot fit 3 of the L-shaped cages into a 3x3 box *(e.g. figure 7.1)* nor can we fit only 2 of the 1x3 cages and 1 L-shaped cage *(e.g. figure 7.2),*

Hence, we can only fit either all 1x3 cages, or 1 1x3 cage and 2 L-shaped cages. Factoring in rotation, there are 3 different ways to tile it. *(Figure 8.1, 8.2, 8.3).*

Including Rotation

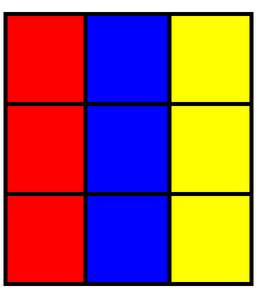
Figure 8.1

X2

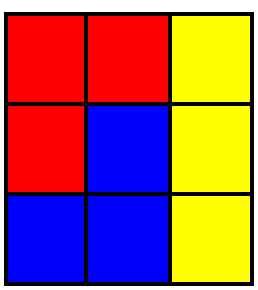
Figure 8.2

X4

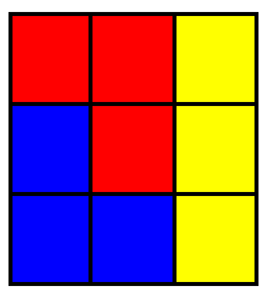
Figure 8.3

X4

If we include rotations, there will be 2+4+4 which is 10 possibilities.

Since there are nine 3x3s in one 9x9 sudoku, the answer will be 10^9, which is 1000000000, a billion.

4.2 Using multiplication, subtraction and addition cages

4.2.1 On a 4x4 Grid

This is the full question:

How many ways are there to arrange 1×2 cages in killer sudokus, with at least 1 cage each containing multiplication, subtraction and addition for a 4x4 killer sudoku?

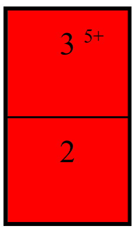

Figure 9.1

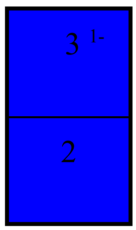

Figure 9.2

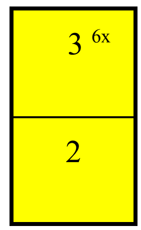

Figure 9.3

Figures 9.1, 9.2 and 9.3 show the 3 different types of cages. Note that for subtraction, the higher number will subtract the lower number. In this research question, the numbers in the cells are also redundant so we will be representing addition as red, subtraction as blue, and multiplication as yellow for convenience.

From Research Question 1, we know that there are **36** possible arrangements of 1x2 cages in 4x4 killer sudokus. Now, we can just focus on 1 of the possible arrangements, then multiply the result by 36 to get the answer. The arrangement we will be using is shown in figure 10.

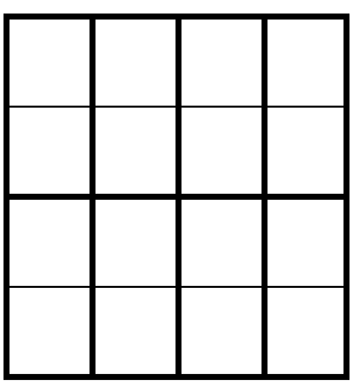
Figure 10

In each of the 1x2 cages, it can contain 3 different colours, red, blue or yellow. Therefore, the number of possibilities for the arrangements of colours is 3^8, since there are 8 1x2 cages.

However, we are not done yet. Remember that we set a condition that there must be all 3 types of cages present? Well, there are 2 cases to explore:

1. 1 type of cage is present only
2. 2 types of cages are present

4.2.1.1 Case 1

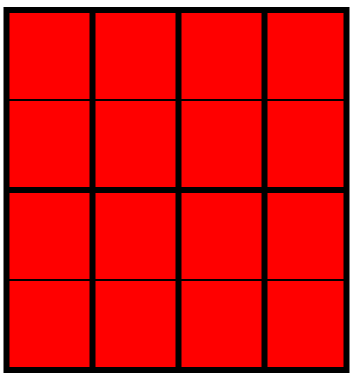

Figure 11.1

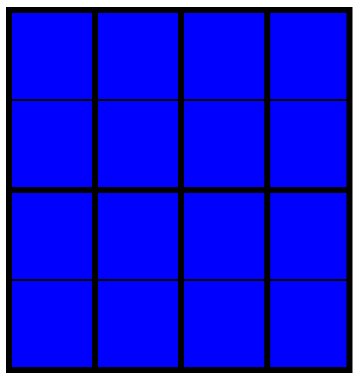

Figure 11.2

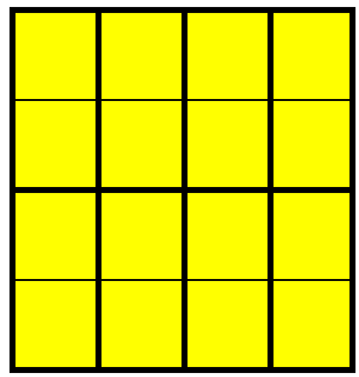

Figure 11.3

As there are only 3 colours, the solution is 3 as shown from Figure 11.1-11.3

4.2.1.2 Case 2

How about if there are 2 types of colours?

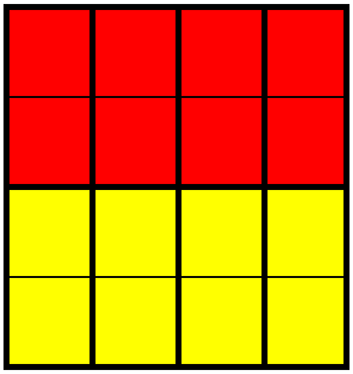

Figure 12.1

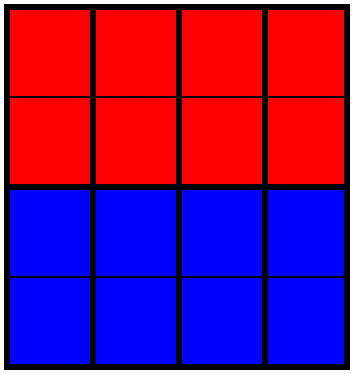

Figure 12.2

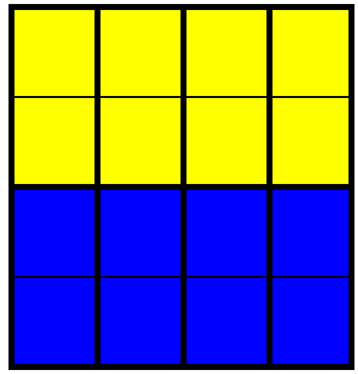

Figure 12.3

There are 3 different colour combinations (Figure 12.1-12.3). However, we will be focussing on figure 12.1 first. Then, we can multiply the results by 3 to get the number of cases with only 2 colors.

Once again, we can split it into several cases:

A. 7 red cages, 1 yellow cage
B. 6 red cages, 2 yellow cages
C. 5 red cages, 3 yellow cages
D. 4 red cages, 4 yellow cages
E. 3 red cages, 5 yellow cages
F. 2 red cages, 6 yellow cages
G. 1 red cage, 7 yellow cages

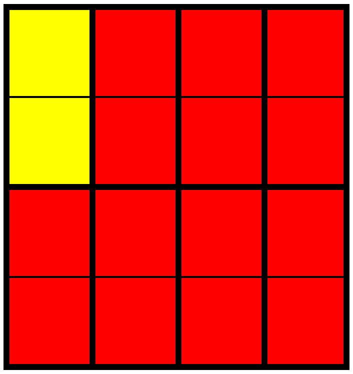

Figure 13.1 (case A)

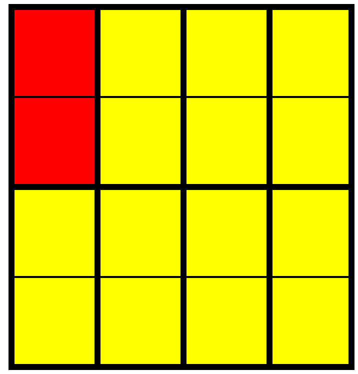

Figure 13.2 (case G)

However, we notice that cases A, B and C have the same number of possibilities as cases G, F and E respectively. This is because the corresponding cases can be obtained by flipping the colours *(e.g. figures 13.1, 13.2)*. As in, the red cages can be swapped with yellow, and the yellow with red. This essentially means that all the arrangements will already be covered by the other case, just in reverse colours.

So, the cases can be shortened to:

A. (7 red cages, 1 yellow cage) x2
B. (6 red cages, 2 yellow cages) x2
C. (5 red cages, 3 yellow cages) x2
D. 4 red cages, 4 yellow cages

While solving this research question, we got a pattern. To show this pattern, we will let k be an operation which means the sum of all the positive integers equal or below the number, and is prioritised before addition.

E.g:

$6k = 6 + 5 + 4 + 3 + 2 + 1$

$3k^2 = (3 + 2 + 1)k \quad = 3k + 2k + 1k$

The pattern is that the final result would be:

(number of red cages + 1)($k^{\text{number of yellow cages - 1}}$)

This pattern was formed based on logical thinking. The “number of red cages + 1” actually refers to the number of areas which the last yellow cage can be at. The “number of yellow cages - 1” refers to the number of yellow cages other than the last yellow cage. Below is our rationale behind this pattern.

Let’s say we number the yellow cages, from 1^{st} to (number of yellow cages)th. Then, we randomly insert the 1st cage first, until the last cage anywhere we want, maybe all at the top row. Then, the last cage will have (number of red cages + 1) areas to be in. This is true for all cases.

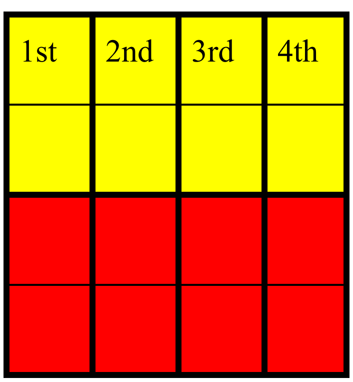

Figure 14.1

Then, the 2nd last yellow cage also has (number of red cages + 1) places to be, and hence (number of red cages) places to move.. However, when it moves to another place, its original space cannot be occupied by the last yellow cage, if not it will be a repeat. Therefore, it is effectively restricting the last yellow cage's possible arrangement by 1. Since it can move (number of red cages) times, so the number of possible arrangements for the 2nd last and last yellow cage is:

(number of red cages + 1) +(number of red cages) + (number of red cages - 1) + ... + (1)

Which is basically:

(number of red cages + 1)k

Figure 14.2 is an example. The blue bordered red cage cannot be filled in with the last yellow cage, if not it is the same as figure 14.1.

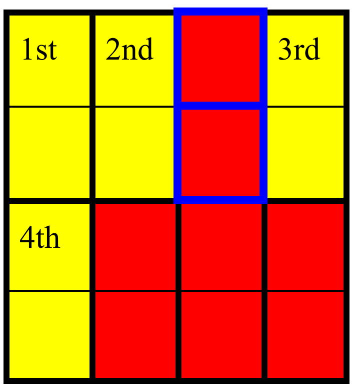

Figure 14.2

This pattern goes on and on. For the 3rd last yellow cage, we can also move it (number of red cages) times, and the 2nd last and last yellow cages cannot be in the space that the 3rd last yellow cage was originally in, if not it would be a repeat. Hence, it also effectively reduced the number of places the 2nd last and last yellow cage can go to by 1. This means that the number of arrangements for the 3rd last, 2nd last and last is:

(number of red cages + 1)k + (number of red cages)k + … + (1)

Which is basically:

(number of red cages + 1)k^2

An example is in figure 14.3 where neither the 3rd nor 4th cage can go back to the original position of the 2nd cage, which is highlighted in blue, if not it would be a repeat of figure 14.1 and 14.2.

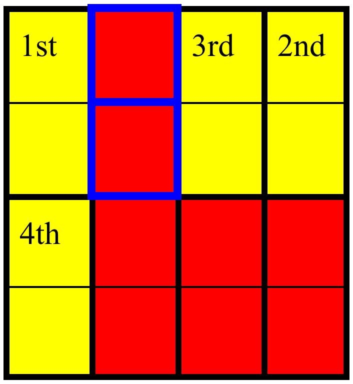

Figure 14.3

Using the same logic, there will be:

(number of red cages + 1)kkk possible arrangements for all 4 of the yellow cages.

An example is figure 14.4, where none of the other cages can go back to the original position of the 1st cage, if not it would be a repeat of the other figures.

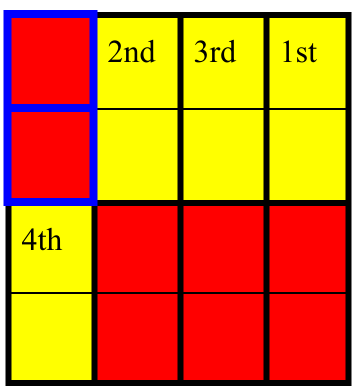

Figure 14.4

After deriving this formula:

(number of red cages + 1)($k^{\text{number of yellow cages - 1}}$), we managed to calculate the possible arrangements of each of the cases and after adding them up, we got 8 + 7k + 6kk + 5kkk = **254**. However, we still needed to multiply it by 3(number of different 2 colour combinations) which gives it **762**

Conclusion

The number of ways to arrange 1×2 cages in killer sudokus, with at least 1 cage each containing multiplication, subtraction and addition for a 4x4 killer sudoku is:

([Number of combinations of multiplication, subtraction and addition cages] - [number of cases with 1 type of cage] - [number of cases with 2 types of cages]) x (number of arrangements of 1x2 cages in a 4x4)

$(3^8 - 3 - 762) \times 36 =$ **208656**

4.2.2 On a 9x9 Grid

Due to the shape of the cages, it will be unclear which number to subtract first as shown is figure 15, subtraction is not allowed for trimino cages and cages with more than 4 cells. As such there's only addition and multiplication, two cases.

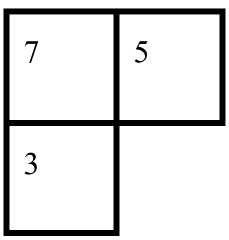

Figure 15

Total number of possibilities for types of cages: 2^{27}

However we need to minus away the cases with only 1 type of cage. As you can see in figure 16.1 and 16.2, there are only 2 such possibilities, either all yellow or all red.

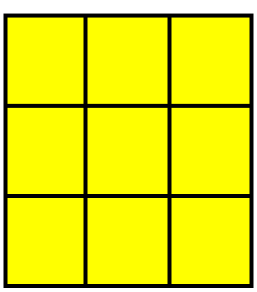

Figure 16.1

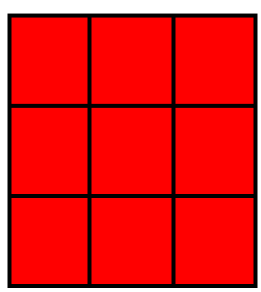

Figure 16.2

Total number of possibilities for types of cages for each trimino arrangement: $2^{27}-2$

However, since there are 10^9 number of trimino arrangements, therefore the total number of arrangements and possibilities is: $(2^{27}- 2) \times 1000000000$ = **134217726000000000**

Or $1.34217726 \times 10^{17}$

4.3 Using triminos and 1x1 cages

4.3.1 On a 9x9 Grid

This time, we will be focussing on the 9x9 only.

To recap, here's the question: *Find the number of ways to arrange cages using only triominos and 1x1 cages in a 9x9 killer sudoku, given that the cages cannot cross the nonet.*

For this, we can set up 4 cases in each nonet:

1. All triminos
2. 2 triminos, 3 1x1 cages
3. 1 trimino, 6 1x1 cages
4. All 1x1 cages

4.3.1.1 Case 1

As done in Research Question 1*(Page 7-8)*, there are **10** possibilities

4.3.1.2 Case 2

There are 3 triminos in the 3x3, and we can break up each trimino into 3 1x1 cages. Therefore, there are 3 different possibilities for each trimino arrangement, and since there are 10 arrangements, the total number of possibilities is 10x3= **30.** An example is shown in figures 17.2, 17.3 and 17.4. It illustrates how each trimino can be broken up from 17.1 into 3 1x1 cages to form unique cages.

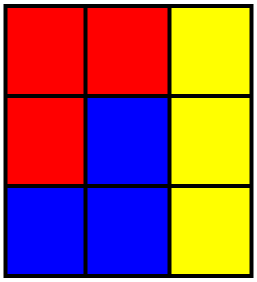
Figure 17.1

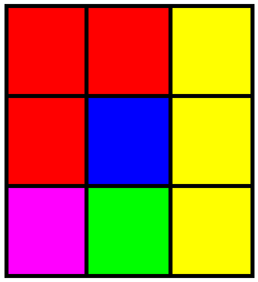
Figure 17.2

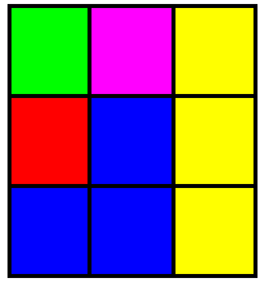
Figure 17.3

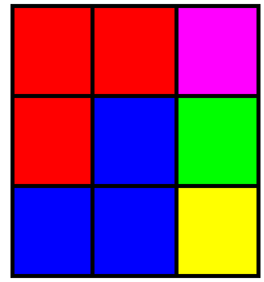
Figure 17.4

4.3.1.3 Case 3

First, we can assume the entire 3x3 is empty, then find the number of arrangements of the trimino, then fill in the 1x1 cages afterwards.

For 1x3 cages, there are **6** different places it can be located at, 3 vertical and 3 horizontal.

For L-shaped cages, since there are 3 possible arrangements for each rotation *(e.g. figure 18.1, 18.2, 18.3)*, so after rotation, there will be 3x4= **12** possibilities.

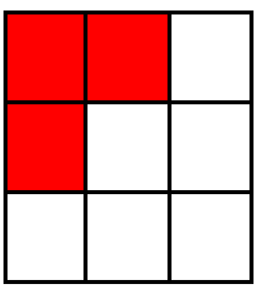
Figure 18.1

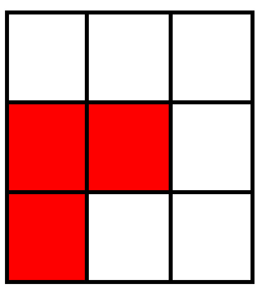
Figure 18.2

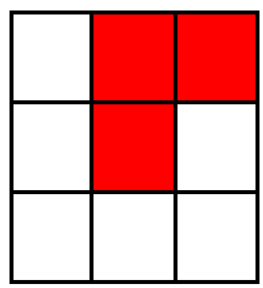
Figure 18.3

Hence, in total, there are 12+6=**18** possible arrangements for 1 trimino

4.3.1.4 Case 4

If all are 1x1 cages, it is quite obvious that there is only **1** possible arrangement.

Conclusion

So, in each nonet, there will be 10+30+18+1=**59** possible arrangements.

However, we have 9 nonets, so the total is 59 to the power of 9, which is **8 662 995 818 654 939.**

5 Extensions

It can be extended to include killer sudokus of different sizes, not just 9x9.

We can also use killer sudokus with other shapes that may be harder to solve.

Division can also be included as one of the possible arrangements.

6 References

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